Sri Chandrasekharendra Saraswathi Viswa Mahavidhyalaya

Graph Theory

Programme course

4 credits

Main field of study

Mathematics

Course offered for

Mathematics, Bachelor´s Programme

Specific information

The course is available every second year

Course overview

This course is an introductory course to the basic concepts of Graph Theory. This includes definition of graphs, vertex degrees, directed graphs, trees, distances, connectivity and paths

Prerequisites

Some elementary knowledge of linear algebra, particularly matrix algebra, would be helpful. In addition, a general experience in mathematics.

Course objectives

- The objective of the course is to introduce students with the fundamental concepts in graph Theory, with a sense of some its modern applications.
- They will be able to use these methods in subsequent courses in the design and analysis of algorithms, computability theory, software engineering, and computer systems

Intended learning outcomes

On successful completion of this course, student should be able to

- understand the basic concepts of graphs, directed graphs, and weighted graphs and able to present a graph by matrices.
- understand the properties of trees
- understand Eulerian and Hamiltonian graphs.
- apply the knowledge of graphs to solve the real-life problem.

Program Outcomes

The students should be able to

- Solve problems using basic graph theory
- To write precise and accurate mathematical definitions of objects in graph theory.
- Use definitions in graph theory to identify and construct examples and to distinguish examples from non-example.
- Determine whether graphs are Hamiltonian and/or Eulerian
- Apply theories and concepts to test and validate intuition and independent mathematical thinking in problem solving.
- Integrate core theoretical knowledge of graph theory to solve problems.
- Reason from definitions to construct mathematical proofs
- Model real world problems using graph theory

Course content

Unit I
Graphs and Subgraphs – Introduction – Definition and Examples – Degree of a vertex – subgraphs – isomorphism of Graphs – Ramsey Numbers – Independent sets and Coverings
Unit-II
Intersection Graphs and Line Graphs – Adjacency and Incidence Matrices – Operations on Graphs – Degree Sequences – Graphic Sequences
Unit III
Connectedness **-**Introduction – Walks, Trails, paths, components, bridge, block - Connectivity
Unit IV
Eulerian Graphs – Hamiltonian Graphs
Unit V
Trees – Characterization of Trees – Centre of a Tree – Planarity – Introduction, Definition and Properties – Characterization of Planar Graphs – Thickness – Crossing and Outer Planarity

Course literature

S.Arumugam and S.Ramachandran, "Invitation to Graph Theory", SCITECH Publications India Pvt. Ltd., 7/3C, Madley Road, T.Nagar, Chennai - 17

Reference Books

1. S.Kumaravelu, SusheelaKumaravelu, Graph Theory, Publishers, 182, Chidambara Nagar, Nagercoil-629 002.

2. S.A.Choudham, A First Course in Graph Theory, Macmillan India Ltd.

3. Robin J.Wilson, Introduction to Graph Theory, Longman Group Ltd.

4. J.A.Bondy and U.S.R. Murthy, Graph Theory with Applications, Macmillon, London.

UNIT – I

Graphs and Sub graphs : Definition and examples of graphs – degrees – sub graphs – isomorphism – Ramsey numbers – independent sets and coverings

1.1 DEFINITION AND EXAMPLES(1).GRAPH

Definition: A graph G consists of a pair (V(G), X(G)) , where V(G) is a non – empty finite set whose elements are called ***points or vertices*** and X(G) is another set of unordered pairs of distinct elements of V(G). The elements of X(G) are called ***lines*** or ***edges*** of the graph.

If $x = \{u, v\} \in$ X then the line x is said to ***join*** of u and v. The points u and v are said to ***adjacent*** if $x = u\ v$. We say that the points u and the line x are incident with each other.

If two distinct lines x and y are incident with a common point then they are called ***adjacent lines***.

A graph with p points and q lines is called a (p, q) graph.

Note: When there is no possibility of confusion we write V (G) = V andX (G) = X.

Examples:

Let V = $\{a, b, c, d\}$ and X = $\{\{a, b\},\{a, c\},\{a, d\}\}$.

G = {V, X} is a (4, 3) graph. This graph can be represented by a diagram as shown in Fig.1.1.

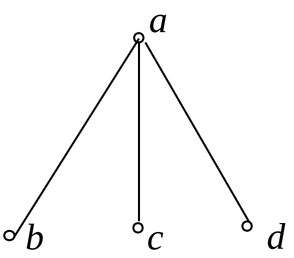

Fig. 1.1

In this graph the points *a* and *b* are adjacent whereas *b* and *c* are non – adjacent.

2. Let V = {1, 2, 3,4} and X ={{1,2},{1,3},{1,4},{2,3},{2,4},{3,4}.

G = (V, X) is a (4, 6) graph.

This graph is represented by the diagram as shown in Fig.1.2.

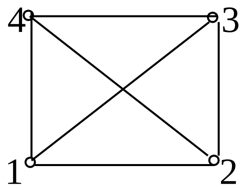

Fig. 1.2

In this graph the lines {1, 3} and {2, 4} intersect in the diagram and their intersection is not a point of the graph.

Fig. 1.3 is another diagram for the graph given in Fig. 1.2.

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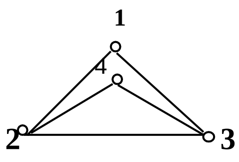


Fig. 1.3

3. The (10, 15) graph given in Fig. 1.4 is called a ***Petersen graph***.

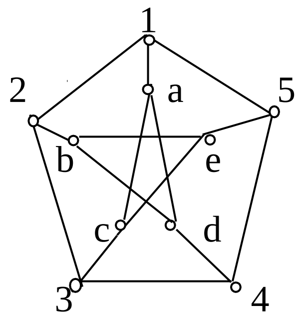


*Fig.*1.4

Remark:

The definition of a graph does not allow more than one line joining two points. Also, it does not allow any line joining a point to itself.

Line joining points to itself is called a ***loop.*** Fig.1.5 is a loop.

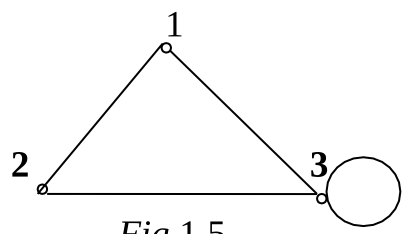


*Fig.*1.5

(2). MULTI GRAPH

Definition: If more than one line joining two vertices are allowed then the resulting object is called a ***multi graph***. Lines joining the same points are called ***multiple lines***.

Fig. 1.6 is an example of a multi graph.

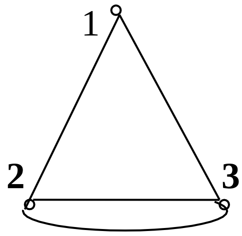


Fig. 1.6

(3). PSEUDO GRAPH

Definition: If an object contains multiple lines and loops then it is called a ***pseudo graph***.

Fig. 1.7 is a pseudo graph.

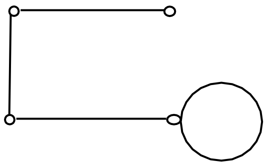


Fig. 1.7

Note: If G is a (*p, q*) graph then $q \leq p\ C_2$ and $q = p\ C_2$ if and only if any two distinct points are disjoint.

(4).COMPLETE GRAPH

Definition: A graph in which any two distinct points are adjacent is called a ***complete graph***.

A complete graph with *p* vertices is denoted by K_p.

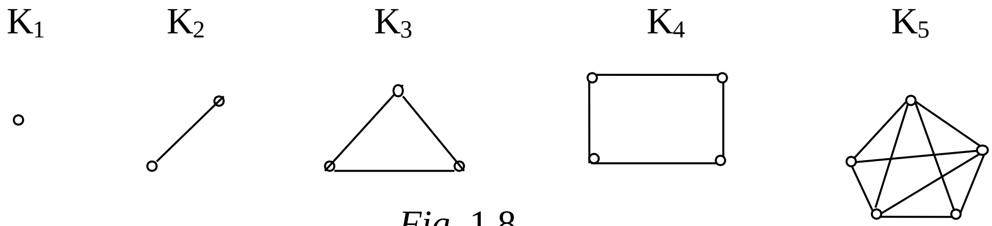


Fig. 1.8

Note: The number of edges of a complete graph K_p is $p\ C_2$.

(5). NULL GRAPH

Definition: A graph G whose edge set is empty is called a ***null graph*** or a ***totally disconnected graph*.**

Example: G_1, G_2, G_3 and G_4 are null graphs.

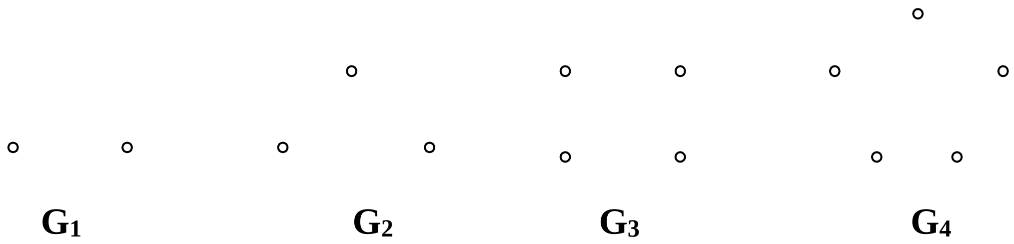


Fig. 1.9

(6).LABELLED GRAPH

Definition: A graph G is called ***labelled*** if its *p* points are distinguished from one another by names such as $v_1, v_2, ..., v_p$.

The graphs given in Fig. 1.1 and Fig.1.2 are labelled graphs and the graph in Fig. 1.8 is an unlabelled graph.

(7).BIPARTITE GRAPH

Definition: A graph G is called a ***bigraph*** or ***bipartite graph*** if the vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every line of G joins a point of V_1 to a point of V_2. (V_1, V_2) is called a ***bipartition*** of G.

(8).COMPLETE BIPARTITE GRAPH

Definition: A graph G is called a ***complete bipartite graph*** if the vertex set V can be portioned into two disjoint subsets V_1 and V_2 such that every line joining the points of V_1 to the points of V_2. If V_1 contains *m* points V_2 contains *n* points then the complete bigraph is denoted by $K_{m, n}$.

The complete graph $K_{1, n}$ is called a ***star*** for $n \geq 1$.

Note: The number of points of the complete bigraph $K_{m,n}$ is $m + n$ and the number of lines is $m\,n$.

Example:

$K_{4,2}$ **$K_{2,3}$** **$K_{3,3}$**

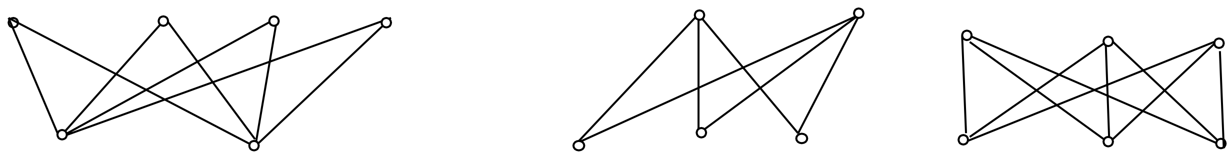


Fig. 1.10

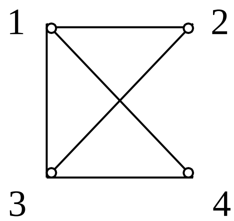
Problem:

Let V = {1, 2, 3, . . . , n} . Let X = {(i , j) / i , j ∈ V and are relatively prime }. The resulting graph (V, X) is denoted by G_n. Draw G_4 and G_5.

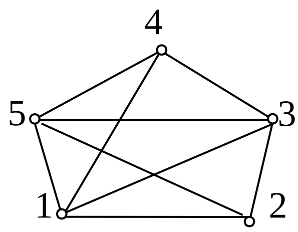
Solution:

For G_4 : V = { 1, 2, 3, 4} and X = { (1,2), (1,3), (1,4), (2, 3), (3, 4)}.

For G_5 : V = { 1, 2, 3, 4,5} and X = {(1,2), (1,3), (1,4), (1,5),(2,3), (2,5), (3, 4), (3, 5), (4, 5)}.



Graph of G_4 (Fig. 1.11*)*



Graph of G_5 (Fig. 1.12*)*

1.2. DEGREES

Definition: The ***degree*** of a point v_i in graph G is the number of lines incident with v_i. It is denoted by $d(v_i)$ or deg v_i or $d_G(v_i)$.

A point v of degree *0* is called an ***isolated point.*** A point v of degree *one* is called an ***end point***.

Note: Loops are counted twice.

Example: Consider the following (4, 5) graph, Fig.1.13**.**

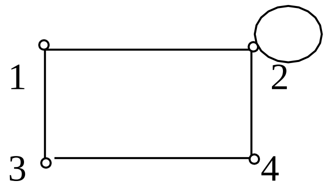


*Fig.*1.13

$d(1) = 2$, $d(2) = 4$, $d(3) = 2$, $d(4) = 2$.

Total degrees = 10 = 2 x 5.

Theorem 1.1: The sum of the degrees of the points of a graph G is twice the number of lines in G. i.e. $\sum d(v_i) = 2q$.

Proof: Every line of G is incident with two points.

$\therefore$ Every line contributes two degrees.

There are q lines in (p, q) graph.

$\therefore \sum_{i=1}^{p} (v_i) = 2q$ = 2x (number of lines in G).

Theorem 1.2: In any group G the number of points of odd degree is even.

Proof: Let $v_1, v_2, \ldots, v_k$ denote the points of odd degree and $w_1, w_2, \ldots, w_m$ denote the points of even degree in G.

By theorem (1.1), $\sum_{i=1}^{k} d(v_i) + \sum_{j=1}^{m} (w_j) = 2q$, which is even.

Also, $\sum_{j=1}^{m}(w_j)$ is even.

$\therefore \sum_{i=1}^{k}(v_i)$ is even.

But, $d(v_i)$ is odd for each i .

Hence, k is even .

∴ the number of points of odd degree is even.

Definition :(REGULAR GRAPH)

For any graph G, we define

δ (G) = min {*d (v)* / *v* ∈ V (G)} and

Δ (G) = max {*d (v)* / *v* ∈ V (G)}.

If all points of G have the same degree *r* then G is called a ***regular graph*** of degree *r*. Hence, in a regular graph δ (G) = Δ (G).

A regular graph of degree *3* is called a ***cubic graph***.

Example(1): Consider the following graph , Fig.1.14**.**

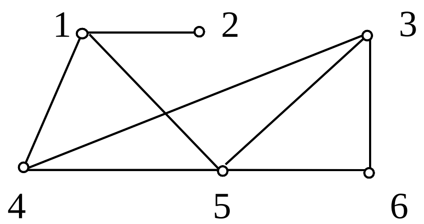


Fig. 1.14.

d(1) = 3, *d*(2) = 1, *d*(3) = 3, *d*(4) = 3, *d*(5) = 4, *d*(6) = 2.

Here, δ = 1, Δ= 4 ⇒ $\delta \neq \Delta$.

∴ The given graph is not regular.

Example (2): Consider the graph as given in Fig.1.2**.**

Here, $d(1) = 3$, $d(2) = 3$, $d(3) = 3$, $d(4) = 3$.

$\therefore \delta = \Delta = 3 \Rightarrow$ the graph is regular.

Example (3):

(i). A null graph is a regular graph of degree 0.

(ii). The complete graph K_p is a regular graph of degree $(p - 1)$.

Theorem 1.3: Every cubic graph has an even number of points.

Proof: Let G be a cubic graph with p points.

To show that p is even.

$\therefore \sum_{i=1}^{p} (v_i) = 3p$, since G is a cubic graph

We know that, by theorem (1.1), $\sum_{i=1}^{p} (v_i)$ is an even number.

$\therefore 3p$ is even $\Rightarrow p$ is even.

Hence, every cubic graph has an even number of points.

SOLVED PROBLEMS

Problem (1): Let G be a (p, q) graph all of whose points have degree k or $k + 1$.

If G has $t > 0$ points of degree k then show that $t = p(k + 1) - 2q$.

Solution:

Given that G is a (p, q) graph and all of whose points have degree k or $k + 1$.

Also, given that G has t points of degree k.

$\therefore$ the remaining $p - t$ points have degree $k + 1$.

We know that, $\sum_{i=1}^{p} d(v_i) = 2q$.

i.e. $t\,k + (\,p - t)\,(\,k + 1) = 2q$

$\Rightarrow\; t\,k + p\,k - t\,k + p - t \;=\; 2q$

$\Rightarrow t = p\,k + p - 2q$

$\Rightarrow\; t = p\,(k + 1) - 2q.$

Problem (2): Show that in any group of two or more people, there are always two with exactly the same number of friends inside the group.

Solution:

Construct a graph G by taking the group of people as the set of points and joining two of them if they are friends.

Then *deg v* = number of friends of *v*.

To prove that at least two points of G have the same degree.

Let $v_1, v_2, \ldots, v_p$ be the points of G , where $p \geq 2$.

Clearly $0 \leq deg\, v_i \leq p - 1$ for each *i*.

Suppose no two points of G have the same degree.

Then the degree of points $v_1, v_2, \ldots, v_p$ are $0, 1, 2, \ldots, p - 1$ in some order.

But, a point of degree (p – 1) is joined to ever other point of G.

Hence, no point can have degree zero. This is a contradiction to the fact that point of G has degree zero.

Thus, there exist two points of G with the same degree.

Problem (3): What is the maximum degree of any point in a graph with p points?

Solution:

Line is obtained by a selection of any two points from the p points.

∴ Maximum number of lines = $pC_2 = p\,(p-1)/2$.

$$\therefore \quad \sum_{i=1}^{p} d(v_i) = p\,(p-1)$$

$$\therefore \quad d\,(v_i) \le (p-1)$$

Hence, the maximum degree of any point in a graph with p points is $(p-1)$.

Problem (4): Prove that $\delta \le \frac{2q}{p} \le \Delta$.

Solution:

Let G be a (p, q) graph and $v_1, v_2, \ldots, v_p$ be the points of G.

We know that, $\sum_{i=1}^{p} (v_i) = 2q$.

Also, $\delta\,(G) = \min \{d\,(v) / v \in V\,(G)\}$ and

$\Delta\,(G) = \max \{d\,(v) / v \in V\,(G)\}$.

$\therefore \delta \le d\,(v_i) \le \Delta$ for all i.

$$\therefore \sum_{i=1}^{p} \delta \le \sum_{i=1}^{p} d(v_i) \le \sum_{i=1}^{p} \Delta .$$

i.e. $p\delta \le 2q \le p\Delta$.

i. e. $\delta \le \frac{2q}{p} \le \Delta$.

Problem (5): Let G be a k – regular bigraph with bipartition (V_1, V_2) and $k > 0$. Prove that $|V_1| = |V_2|$.

Solution:

Given that G is a k – regular bigraph with bipartition (V_1, V_2) and $k > 0$.

We know that, "Every line of G joins a point of V_1 to a point of V_2".

$$\therefore \sum_{v \in V_1} (v) \quad = \sum_{v \in V_2} d(v).$$

Also, $d(v) = k$ for all $v \in V_1 \cup V_2$.

Hence, $\sum_{v \in V_1} k \quad = \sum_{v \in V_2} k.$

i.e. $k\,|V_1| = k\,|V_2|$.

i.e. $|V_1| = |V_2|$, since $k > 0$.

Problem (6): A (p, q) graph has t points of degree m and all other points are of degree n. Show that $(m – n)\,t + p\,n = 2\,q$.

Solution:

Given that G is a (p, q) graph and t points of degree m.

The remaining $(p – t)$ points have degree n.

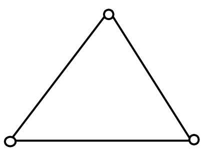
We know that, $\sum_{v \in V} (v) = 2\,q$.

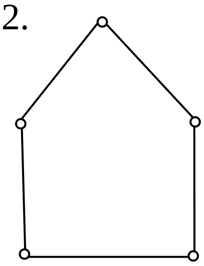
i.e. $m\,t + (p - t)n = 2q$.

i.e. $(m – n)\,t + p\,n = 2\,q$.

Problem (7): Give three examples for regular graph of degree 2.

Solution:





Exercises:

1. Give an example of a regular graph degree 0.
2. Give an example of a regular graph degree 1.
3. Give three examples for a cubic graph.
4. If G is a graph with at least two points then show that G contains two vertices of the same degree.
5. Show that a graph with p points is regular of degree $p - 1$ iff it is complete.

1.3 SUBGRAPHS

Definition: A graph $H = (V_1, X_1)$ is called a ***subgraph*** of G (V, X)

if $V_1 \subseteq V$ and $X_1 \subseteq X$.

H is a subgraph of G then we say t hat G is a ***supergraph*** of H.

H is called a ***spanning graph*** of G if $V_1 = V$.

H is called an ***induced subgraph*** of G if H is the maximal subgraph of G with point set V_1.

i.e. if H is an induced subgraph of G then two points are adjacent in H if and only if they are adjacent in G.

Example (1): Consider the graph G given in Fig. 1.15.

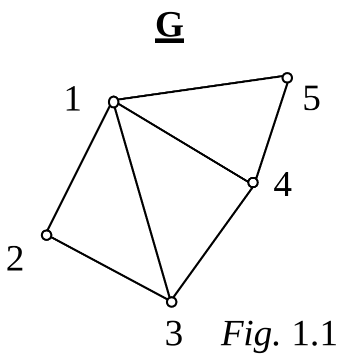
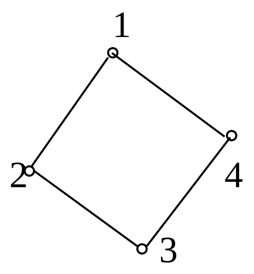


Fig. 1.15

Subgraph , H_1



Subgraph, H_2

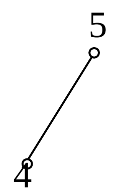


Fig. 1.16

Spanning Subgraph

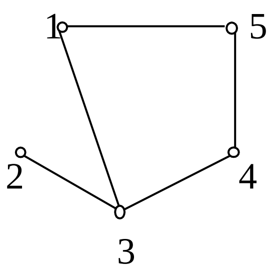


Fig. 1.17

Induced Subgraph

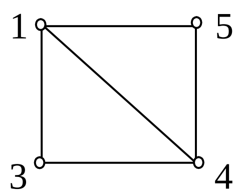


Fig. 1.18

Example (2): Consider the Peterson graph G given in Fig. 1.4.

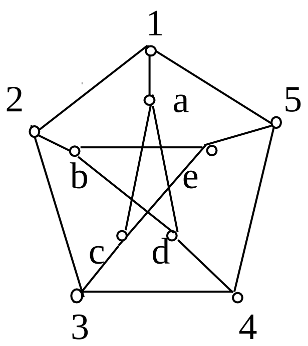


*Fig.*1.4

Subgraph of G

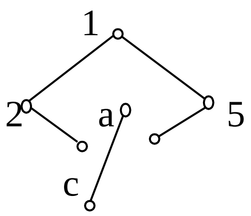


Fig. 1.19

Induced subgraph of G

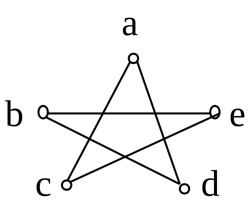


Fig. 1.20

Spanning Subgraph of G

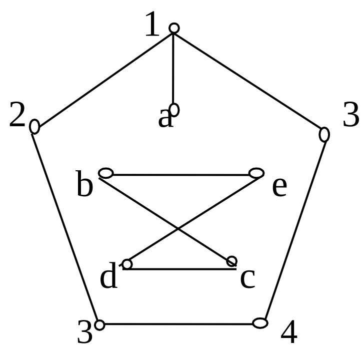


Fig. 1.21

REMOVAL OF A POINT

Definition: Let G = (V, X) be a graph and $v \in$ V. The subgraph of G obtained by removing the point v and all the lines incident with v is called the ***subgraph obtained by the removal of the point v*** and is denoted by $G - v$.

i.e. If $G - v = (V_1, X_1)$ then $V_1 = V - \{v\}$ and

$X_1 = \{ x / x \in X$ and x is not incident with $v \}$.

i.e. $G - v$ is an induced subgraph of G.

REMOVAL OF A LINE

Definition: Let G = (V, X) be a graph and $x \in$ X. Then $G - x = (V, X - \{x\})$ is called the ***subgraph of G obtained by the removal of the line x*** .

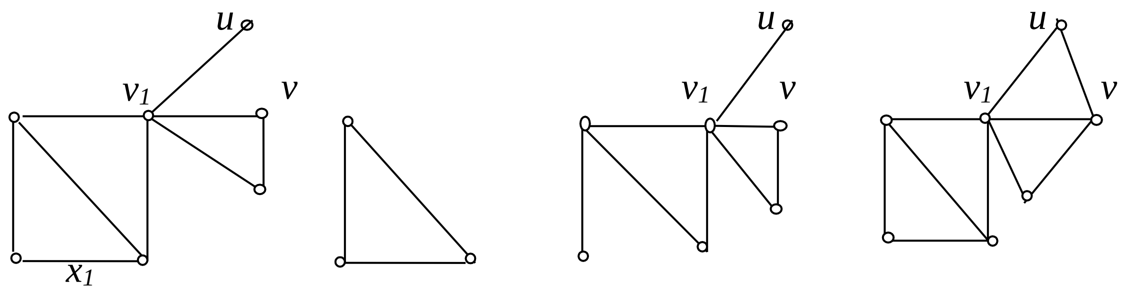
i.e. $G - x$ is a spanning subgraph of G which contains all the lines of G except the line x.

ADDITION OF A LINE

Definition: Let G = (V, X) be a graph. Let u , v be two non adjacent points of G. Then $G + uv = (V, X \cup \{u, v\})$ is called the graph obtained by ***the addition of the line*** uv to G.

i.e. $G + uv$ is the smallest super graph of G containing the line uv.

Example:



G **G – v_1** **G – x_1** **G + uv**

Fig. 1.22

Theorem 1.4: The maximum number of lines among all p point graph with no triangles is $[\frac{p^2}{4}]$, where $[x]$ denotes the greatest integer not exceeding the real number x.

Proof:

The result can be easily verified for $p \leq 4$.

For $p > 4$, we prove by induction separately for odd p and for even p.

Case (1): For odd p.

Clearly the result is true when $p = 1$ or 3.

Assume that the result is true for $p = 2n + 1$.

To prove the result for $p = 2n + 3$.

Let G be a (p, q) graph with $p = 2n + 3$ and has no triangles.

If $q = 0$, then $0 \leq [\frac{p^2}{4}]$.

Let $q > 0$.

$\therefore$ There exist two adjacent points in G.

Let u and v be a pair of adjacent points in G.

Consider the subgraph $G^1 = G - \{u, v\}$.

Then G^1 has $2n + 1$ points and no triangles.

Hence by induction hypothesis,

$$\text{Lines of } G^1 \text{ is } q(G^1) \leq \left[\frac{(2n+1)^2}{4}\right] = \left[\frac{4n^2+4n+1}{4}\right]$$

$$= \left[n^2 + n + \frac{1}{4}\right] = n^2 + n.$$

i.e. $q(G^1) \leq n^2 + n$ ....(1)

Since G has no triangles, no points of G^1 can be adjacent to both u and v in G...(2)

Maximum number of lines between G^1 and u or v is $2n + 1$.

Now, lines in G are of three types.

(i). Lines of G^1 [$\leq n^2 + n$, by (1)]

(ii). Lines between G^1 and $\{u, v\}$ [$\leq 2n + 1$, by (2)]

(iii). Line uv .

Hence, Line of G , $q(G) \leq q(G^1) + (2n + 1) + 1$.

$$\leq n^2 + n + 2n + 2$$

$$= n^2 + 3n + 2$$

$$= \frac{4n^2 + 12n + 8}{4}$$

$$= \frac{4n^2 + 12n + 9 - 1}{4}$$

$$= \frac{(2n+3)^2}{4} - \frac{1}{4}$$

$$= [\frac{(2n+3)^2}{4}]$$

i.e. q (G) $\leq [\frac{p^2}{4}]$, where $p = 2n + 3$.

∴ The result is true for all odd p.

Also for $p = 2n + 3$, the graph $K_{n+1, k+2}$ has no triangles and the number of lines is $q = (n+1)(n+2)$

$$= n^2 + 3n + 2$$

$$= \frac{4n^2 + 12n + 8}{4}$$

$$= [\frac{(2n+3)^2}{4}]$$

$$= [\frac{p^2}{4}].$$

Hence this maximum q is attained.

Case (2): For even p.

Clearly the result is true when p = 2 or 4.

Assume that the result is true for $p = 2n$.

To prove the result for $p = 2n + 2$.

Let G be a (p, q) graph with $p = 2n + 2$ and has no triangles.

Let $q > 0$.

∴ There exist two adjacent points in G.

Let u and v be a pair of adjacent points in G.

Consider the subgraph $G^1 = G - \{u, v\}$.

Then G^1 has $2n$ points and no triangles.

Hence by induction hypothesis,

Lines of G^1 is $q(G^1) \leq [\frac{(2n)^2}{4}] = [\frac{4n^2}{4}] = n^2$

i.e. $q(G^1) \leq n^2$ .... (1).

Since G has no triangles, no points of G^1 can be adjacent to both u and v in G .... (2)

Maximum number of lines between G^1 and u or v is $2n$.

Now, lines in G are of three types.

(i). Lines of G^1 $[\leq n^2$, by (1)]

(ii). Lines between G^1 and $\{u, v\}$ $[\leq 2n$, by (2)]

(iii). Line uv .

Hence, Line of G is $q(G) \leq q(G^1) + (2n + 1)$.

$$\leq n^2 + 2n + 1$$

$$= n^2 + 2n + 1$$

$$= \frac{4n^2 + 8n + 4}{4}$$

$$= \frac{(2n+2)^2}{4}$$

$$= [\frac{(2n+2)^2}{4}]$$

i.e. q (G) $\leq [\frac{p^2}{4}]$, where $p = 2n + 2$.

∴ The result is true for all even p.

Also for $p = 2n + 2$, the graph $K_{n+1, k+1}$ has no triangles and the number of lines is $q = (n+1)(n+1)$

$$= n^2 + 2n + 1$$

$$= \frac{4n^2 + 8n + 4}{4}$$

$$= [\frac{(2n+2)^2}{4}]$$

$$= [\frac{p^2}{4}].$$

Thus, for $p = 2n + 2$, $K_{n+1, k+1}$ is a $(p, [\frac{p^2}{4}])$ graph without triangles.

∴ q attained its maximum $[\frac{p^2}{4}]$.

Hence the theorem.

1.4 ISOMORPHISM

Definition: Two groups $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ are said to be ***isomorphic*** if there exists a bijection f :$V_1 \rightarrow V_2$ such that $u, v \in V_1$ are adjacent in G_1 if and only if $f(u), f(v) \in V_2$ are adjacent in G_2.

If G_1 is isomporphic to G_2 then we write $G_1 \cong G_2$. The map f is called an isomorphism from G_1 to G_2.

Example (1): The two graphs given in Fig. 1. 23 are isomorphic.

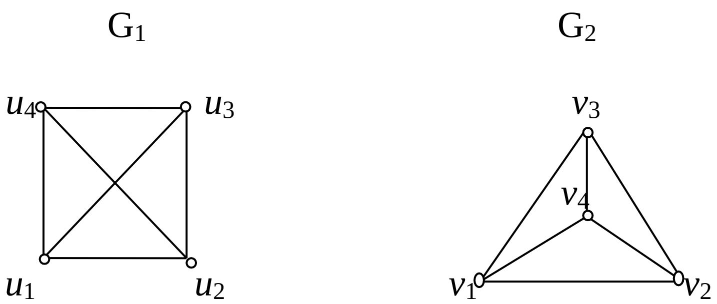


Fig. 1. 23

f (u_i) = v_i is an isomorphism between two groups G_1 and G_2 for i = 1, 2, 3, 4 .

Example (2): The two graphs given in Fig. 1. 24 are isomorphic.

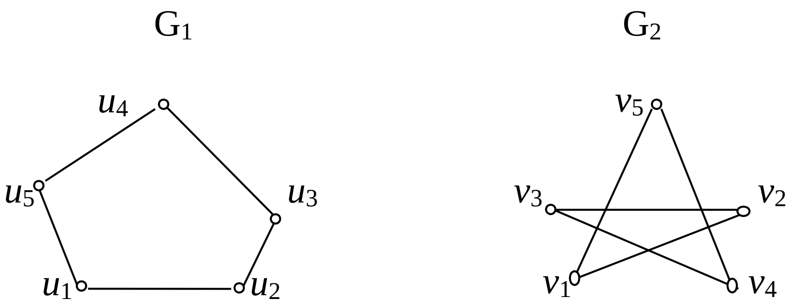


Fig.1. 24

f(u_1) = v_1 , f(u_2) = v_2 , f (u_3) = v_3 , f(u_4) = v_4, f(u_5) = v_5

i.e. f (u_i)= v_i is an isomorphism between two groups G_1 and G_2 for i =1, 2,...,5

Theorem 1.5: Let *f* be an isomorphism of the group $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$. Let $v \in V_1$. Then deg v = deg $f(v)$.

i.e. isomorphism preserves the degree of vertices.

Proof:

Let *f* be an isomorphism of the group $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$.

Given that $v \in V_1$.

∴ a point $u \in V_1$ is adjacent to v in G_1 if and only if f(u) is adjacent to f(v) in G_2.

Also, f is a bijection.

Hence the number of points in V_1 which are adjacent to v is equal to the number of points in V_2 which are adjacent to f(v).

∴ deg v = deg $f(v)$.

Hence isomorphism preserves the degree of vertices.

Remark:

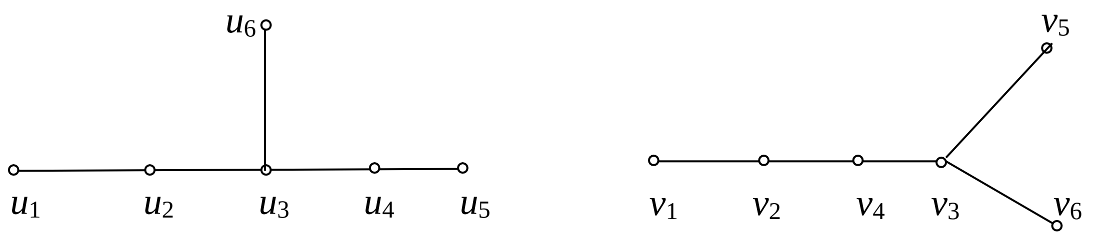
1). Two isomorphic group have the same number of points and the same number of lines.

2). The converse of the above theorem is not true.

i.e. If the degrees of the vertices of two graphs are equal then the two graphs need not be isomorphic.

Example:

Consider the two graphs given in Fig. 1.25.



G_1 *Fig.* 1.25 **G_2**

Here, deg u_i = deg v_i for i = 1, 2, 3, 4, 5, 6.

But G_1 and G_2 are not isomorphic, because u_2 is adjacent to u_3 in G_1 but v_2 is not adjacent to v_3 in G_2.

AUTOMORPHISM

Definition: An isomorphism of a group onto itself is called an ***automorphism*** of G.

Remark:

The set of all automorphism of G is a group. This group is denoted by Γ(G) and is called the ***automorphism group*** of G.

COMPLEMENT

Definition: Let G = (V, X) be a group. The ***complement*** $\bar{G}$ of G is defined to be the graph which has V as its set of points and two points are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

The graph G is said to be a ***self complementary*** graph if G is isomorphic to $\bar{G}$.

Example:

The graphs given in Fig. 1.26 and Fig. 2.27 are self complementary graphs.

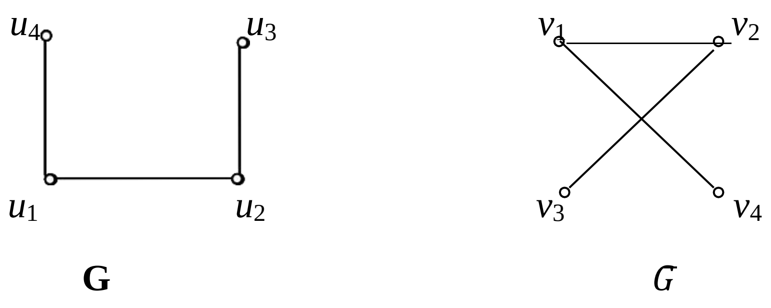


Fig. 1.26

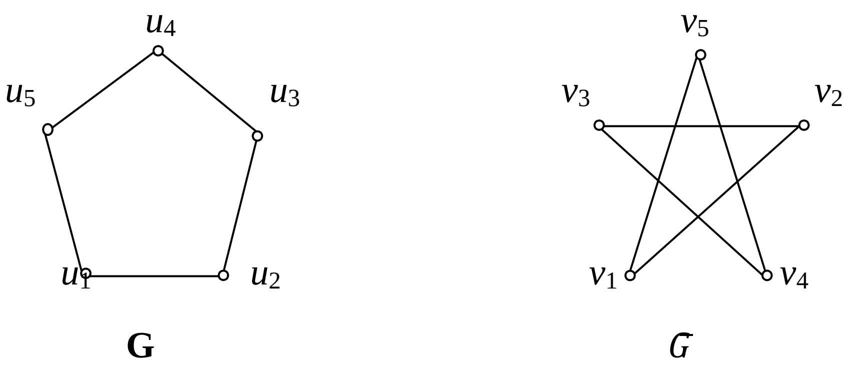


Fig. 1. 27

ULAM’S CONJECTURE

Let G and H be two graphs with p points where $p > 2$. Let $v_1, v_2, \ldots, v_p$ be the points of G and $w_1, w_2, \ldots, w_p$ be the points of H. If for each i the subgraphs $G_i = G - v_i$ and $H_i = H - w_i$ are isomorphic then the graphs G and H are isomorphic.

Ulam"s Conjecture is also known as ***reconstruction conjecture***.

SOLVED PROBLEMS:

Problem 1: Prove that any self complementary graphs has 4 n or 4n + 1 points.

Solution:

Let G be a complementary graph with p points.

$\therefore \quad G \cong \bar{G}$.

$\Rightarrow |X(G)| = |X(\bar{G})|$.

Also, $|(G)| + |X(\bar{G})|$ = Number of edges in K_p .

$$= \binom{p}{2} = \frac{(p-1)}{2}.$$

i.e. $2\,|(G)| \quad = \frac{p(p-1)}{2}.$

$\Longrightarrow \ |X(G)| = \frac{(p-1)}{4}$ is an integer.

i.e. $p\,(p-1) = 4n$, where $n \in Z$.

i.e. p or $(p - 1)$ is a multiple of 4.

$\Longrightarrow \ p = 4n$ or $p - 1 = 4n$.

i.e. $p = 4n$ or $p = 4n + 1$.

Hence, G has $4n$ or $4n + 1$ points.

Problem 2: **P**rove that $\Gamma(G) = \Gamma(\bar{G})$.

Solution:

We know that, $\Gamma(G)$ is a group automorphism of G.

First, we prove that, $\Gamma(G) \subseteq \Gamma(\bar{G})$.

Let $f \in \Gamma(G) \Longrightarrow f : G \rightarrow G$ is an isomorphism.

Let $u, v \in V(G)$.

Now, u, v are adjacent in $\bar{G} \Leftrightarrow u, v$ are not adjacent in G.

- $f(u), f(v)$ are not adjacent in G ,

since f is an automorphism of G.

- $f(u), f(v)$ are adjacent in $\bar{G}$.

$\therefore \ f : \bar{G} \rightarrow \bar{G}$ is an automorphism.

$\therefore f \in \Gamma(\bar{G})$.

Hence, $\Gamma(G) \subseteq \Gamma(\bar{G})$.

Similarly, we can prove that $\Gamma(\bar{G}) \subseteq \Gamma(G)$.

$\therefore \Gamma(G) = \Gamma(\bar{G})$.

Exercises:

1. Show that isomorphism is an equivalent relation among graphs.
2. Prove that any group with p points is isomorphic to K_p.
3. Give a self complementary graph having five vertices.
4. Show that the graphs given in Fig. 1.28 are not isomorphic.

Fig. 1.28

5. Find the complements of the graph given in Fig. 1.24.

1.5 RAMSEY NUMBER

Consider the following puzzle. In any set of six points there will always be either a subset of three who are mutually acquainted, or a subset of three who are mutually strangers. This situation may be represented by a graph G with six points representing the six people in which adjacency indicates acquaintances. The above puzzle asserts that G contains three mutually adjacent points or three mutually non – adjacent points. That is G or $\bar{G}$ contains a triangle.

Theorem 1.6: For any graph G with 6 points, G and $\bar{G}$ contains a triangle.

Proof:

Let G be a group with 6 points.

Let v be a point of G.

Since G contains 5 points other than v , v must be either adjacent to three points in G or non – adjacent to three points in G.

Hence v must be adjacent to three points in G or in $\bar{G}$.

Without loss of generality, let us assume that v is adjacent to three points u_1, u_2, u_3 in G.

If two of these three points are adjacent then G contains a triangle.

Otherwise these three points form a triangle in $\bar{G}$.

Hence G or $\bar{G}$ contains a triangle.

Note: The above theorem is not true for graphs with less than six points.

RAMSEY NUMBER:

Ramsey number is the least positive integer $r(m, n)$ such that for any group G with $r(m, n)$ points , G contains K_m or $\bar{K}_n$.

Example: $r(3, 3) = 6$

$r(1, k) = r(k, 1) = 1$ for any positive integer k.

SOLVED PROBLEMS:

Problem 1: Prove that r (m, n) = r (n, m).

Solution:

Let r (m, n) = s.

Let G be any group with s points. Then $\bar{G}$ also has s points.

Since r (m, n) = s , $\bar{G}$ contains K_m or $\bar{K}_n$.

$\therefore$ G contains $\overline{K_m}$ or K_n.

i.e. G contains K_n or $\overline{K_m}$.

Thus an arbitrary graph on s points contains K_n or $\overline{K_m}$ as an induced subgraph.

$\therefore$ $r(m, n) \leq r(n, m)$ ....(1).

Interchanging m and n, we get

$r(n, m) \leq r(m, n)$ ....(2).

Hence from (1) and (2),

$r(m, n) = r(n, m)$.

Problem 2: Prove that $r(2, 2) = 2$

Solution:

Let G be a graph with 2 points.

Let the two points be u and v.

Then u and v are either adjacent in G or adjacent in $\bar{G}$.

i.e. G or $\bar{G}$. contains K_2.

Thus if G is any graph on two points then G contains K_2 or $\overline{K_2}$.

Clearly 2 is the least positive integer with this property.

$\therefore$ $r(2, 2) = 2$.

Exercises:

1. Prove by suitable example that the theorem 1.6 is not true for graphs with less than 6 points.
2. Find $r(1, 1)$.

3. Find $r(2, 3)$.

4. Find $r(2, k)$ for any positive integer k.

1. 6 INDEPENDENT SETS AND COVERINGS

INDEPENDENCE SET

Definition: Let G = (V, X) be a graph. A subset S of V is called an ***independent set*** of G if no two vertices of S are adjacent in G.

An independent set S is said to be ***maximum*** if G has no independent set S^1 with $|S^1| > |S|$.

The number of vertices in a maximum independent set is called the ***independent number*** of G and is denoted by α.

Example: Consider the graph given in Fig. 1.29.

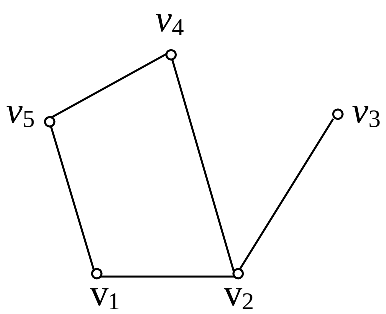


Fig. 1.29

$S_1 = \{v_1, v_3, v_4\}$, $S_2 = \{v_2, v_5\}$, $S_3 = \{v_3, v_4\}$ are independent sets.

S_1 is the maximal independent set.

$$\therefore \ \alpha = |S_1| = 3.$$

VERTEX COVERING

Definition: A ***covering*** of a graph G = (V, X) is a subset K of V such that every line of G is incident with the vertex in K.

A covering K is called a ***minimum covering*** if G has no covering K^1 with $|K^1| < |K|$.

The number of vertices in a minimum covering of G is called the ***covering number*** of g and is denoted by β.

Example: Consider the graph given in Fig. 1. 29.

$K_1 = \{ v_1, v_3, v_4\}$, $K_2 = \{ v_2, v_5 \}$, $K_3 = \{ v_4, v_2, v_5 \}$ are covering of G.

K_2 is the minimum covering.

$\beta = |K_2| = 2$

Theorem 1.7: A set $S \subseteq V$ is an independent set G if and only if V - S is a covering of S.

Proof:

Let G = (V, X) be a graph.

By definition, A set $S \subseteq V$ is an independent iff no two vertices of S are adjacent.

i.e. iff every line of S is incident with at least one point of V - S.

i.e. iff V - S is a covering of G.

Corollary 1.1: For any graph G of p vertices $\alpha + \beta = p$

Proof: Let G be a group with p vertices.

Let S be a maximum independent set and K be a minimum covering of G.

$\therefore \; |S| = \alpha$ and $|K| = \beta$.

By theorem 1.7, S is an independent set iff V – S is an covering.

But, K is a minimum covering of G.

Hence, $|K| \leq |V - S|$.

i.e. $\beta \leq p - \alpha$

i.e. $\beta + \alpha \leq p$.....................................(1)

Also K is a covering iff V – K is an independent set.

But, S is a maximum independent set.

$\therefore \; |V - K| \leq |S|$.

i. e. $p - \beta \leq \alpha$

i. e. $p \leq \alpha + \beta$.....................................(2)

From equations (1) and (2), we get $\alpha + \beta = p$.

LINE COVERING

Definition: A ***line covering*** of a graph G = (V, X) is a subset L of X such that every vertex is incident with a line of L.

The number of lines in a minimum line covering of G is called the ***line covering number*** of G and is denoted by β^1.

A set of lines is called ***independent*** if no two of them are adjacent.

The number of lines in a maximum independent set of lines is called the ***edge independent number*** and is denoted by α^1.

Example: Consider the graph of Fig.1. 30.

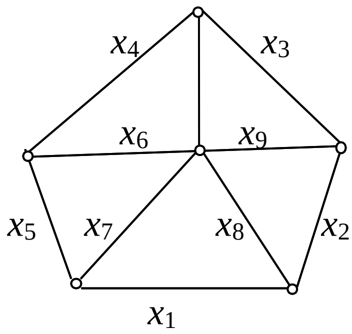


*Fig.*1.30

$L_1 = \{ x_1, x_3, x_6 \}$, $L_2 = \{ x_2, x_4 \}$, $L_3 = \{ x_3, x_5, x_8 \}$ are edge independent sets.

Here, L_1 and L_3 are maximum edge independent sets.

$\therefore\ \alpha^1 = |L_1| = |L_3| = 3.$

$K_1 = \{ x_1, x_3, x_6 \}$ and $K_2 = \{ x_3, x_5, x_8 \}$ are edge covering sets.

i.e. K_1 and K_2 are minimum edge covering sets.

$\therefore\ \beta^1 = |K_1| = |K_2| = 3.$

GALLIA'S THEOREM

Theorem 1.8: For any non – trivial graph with p vertices, $\alpha^1 + \beta^1 = p$.

Solution:

Let G be a (p, q) graph.

We know that, α^1 is the maximal edge independent set and β^1 is the minimum edge covering number.

Let S be the maximum independent set of lines of G.

$\therefore \ |S| = \alpha^1$.

Let M be a set of lines, one incident line for each of the $p - 2\alpha^1$ points of G not covered by any line of S.

Clearly, S U M is a line covering of G.

$\therefore \ |S \cup M| \geq \beta^1$

i.e. $\alpha^1 + p - 2\alpha^1 \geq \beta^1$

i.e. $p \geq \alpha^1 + \beta^1$ ....(1).

Let T be the minimum edge covering set.

$\therefore \ |T| = \beta^1$.

T cannot have a line x, both of whose ends are also incident with lines of T other than x.

Hence G[T], the spanning subgraph of G induced by T is the union of stars.

Hence each line of T is incident with at least one end point of G [T].

Let W be the set of end points of G [T] consisting of exactly one end point for each line of T.

$\therefore \ |W| = |T| = \beta^1$

Hence, $p = |T|$ + number of stars in G [T]

i.e. $p = \beta^1$ + number of stars in G [T] .... (2)

By choosing one line from each star, we get a set of independent lines.

$\therefore \ \alpha^1 \geq$ (number of stars in G[T]) i.e. $p \leq \beta^1 + \alpha^1$ .... (3)

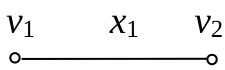
Hence, from equations (1) and (3), $\alpha^1 + \beta^1 = p$.

SOLVED PROBLEMS:

Problem 1: Find α, β, α^1 and β^1 for the complete graph K_p.

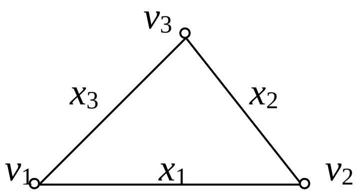
Solution:

K_2



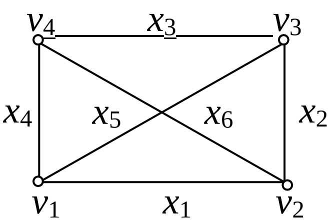
$\alpha = 1, \quad \beta = 1$

$\alpha^1 = 1, \quad \beta^1 = 1$



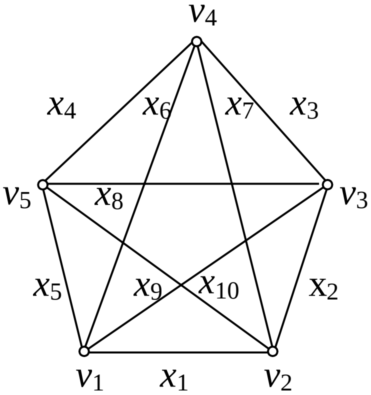
$\alpha = 1, \quad \beta = 2$

$\alpha^1 = 1, \quad \beta^1 = 2$



$\alpha = 1, \quad \beta = 3$

$\alpha^1 = 2, \quad \beta^1 = 2$



$\alpha = 1, \quad \beta = 4$

$\alpha^1 = 2, \quad \beta^1 = 3$

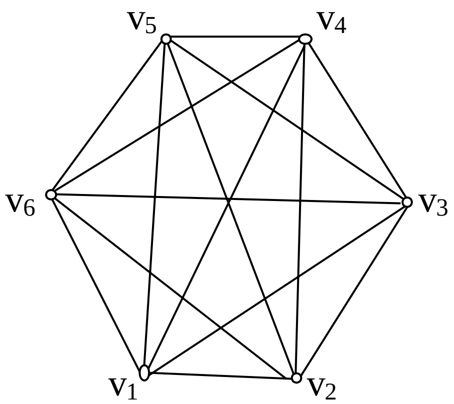


*Fig.*1.31

$\alpha = 1, \quad \beta = 5$

$\alpha^1 = 3, \quad \beta^1 = 3$

$\alpha = 1, \quad \beta = p - 1$

$$\alpha^1 = \begin{cases} \frac{p}{2} & if \ p \ is \ even \\ \frac{p-1}{2} & if \ p \ is \ odd \end{cases} \qquad \text{and} \quad \beta^1 = p - \alpha^1.$$

Problem 2: Give an example to show that the complement of an independent set of lines need not be a line covering.

Solution: Consider the graph given in fig. 1.32.

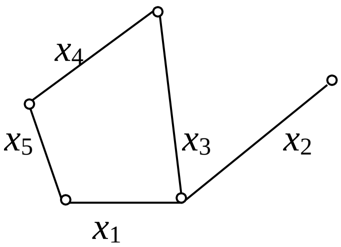


*Fig.*1.32

Independent set, S = { x_2 , x_5}

Complement of independent set, S^1 = { x_1 , x_3, x_4}.

Exercises:

1. Give an example to show that the complement of a line covering need not be an independent set of lines.

2. Prove or disprove. Every covering of a graph contains a minimum covering.
3. Prove or disprove. Every independent set of lines is contained in a maximum independent set of lines.

Unit II

Intersection graphs and line graphs – matrices – operations in graphs – degree sequences, graphicsequences.

INTERSECTION GRAPHS

Definition: Let F = $\{S_1, S_2, \ldots, S_n\}$ be a non - empty family of distinct non – empty subsets of a given set S. The ***intersection graph*** of F, denoted by Ω (F) is defined as follows:

The set of points V of Ω (F) is F itself and two points S_i , S_j are adjacent if $i \neq j$ and $S_i \cap S_j \neq \varphi$.

A graph G is called an intersection graph on S if there exists a family F of subsets of S such that G is isomorphic to Ω (F).

Example:

Let S = { a, b, c } and

F = { {a} , {c} , { a , b} , {b, c}} = { A , B, C, D } , say.

Ω (F)

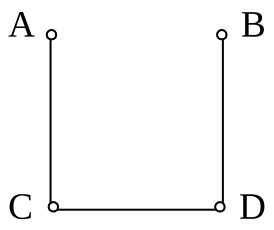


*Fig.*2.1

Theorem 2.1: Every graph is an intersection graph.

Proof:

Let G = (V, X) be a graph.

Let the vertex set V = $\{v_1, v_2, \ldots, v_p\}$.

Let S = V U X.

For each $v_i \in V$, we define

$$S_i = \{v_i\} \cup \{ x \in X \,/\, \text{the edge } x \text{ is incident on } v_i\}.$$

Clearly, F = $\{S_1, S_2, \ldots, S_p\}$ is a family of non – empty subsets of S.

To prove that G is an intersection graph.

i.e. to prove $G \cong \Omega(F)$.

Now, If v_i and v_j are adjacent in G then $x = v_i v_j \in S_i$ and $v_i v_j \in S_j$.

$\Rightarrow v_i v_j \in S_i \cap S_j$

$\Rightarrow S_i \cap S_j \neq \varphi$

$\Rightarrow S_i$ and S_j are adjacent in $\Omega(F)$.

Conversely, If S_i and S_j are adjacent in $\Omega(F)$

$\Rightarrow S_i \cap S_j \neq \varphi$

$\Rightarrow$ The element common to S_i and S_j is the line joining v_i and v_j.

$\Rightarrow v_i$ is adjacent to v_j in G .

Thus the map $f : V \rightarrow F$ defined by $f(v_i) = S_i$ is an isomorphism of G to $\Omega(F)$.

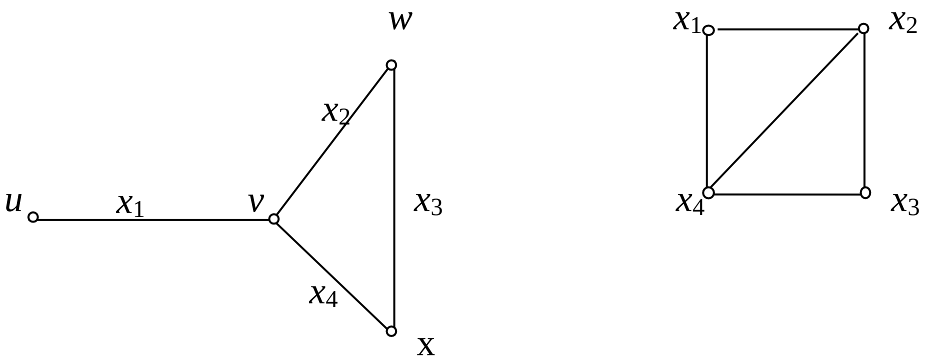
$\therefore$ G $\cong \Omega$ (F).

Hence, every graph is an intersection graph.

LINE GRAPH

Definition: Let G = (V, X) be a graph with X $\neq \varphi$. Then X is a family of two element subsets of V. The intersection graph Ω (X) is called a ***line graph*** of G and is denoted by L(G). Thus the points of L(G) are the lines of G and two points in L(G) are adjacent iff the corresponding lines are adjacent in G.

Example: Consider the graph given in Fig. 1.34.



G **L(G)**

*Fig.*2.2

Theorem 2.2: Let G be a (p , q) graph .Then L(G) is a (q , q_L) graph where $q_L = \frac{1}{2} (\sum_{i=1}^{p} d_i^2) - q$, where d_i is the degree of the vertex v_i in G.

Proof:

Let G be a (p , q) graph.

By the definition of line graph,

Number of points in L(G) is the number of lines in L(G).

∴ L(G) has q points.

Also, d_i is the degree of the vertex v_i in G.

But, Any two of the d_i lines incident with v_i are adjacent in L(G).

∴ $\frac{d_i\ (d_i-1)}{2}$ lines in L(G) .

Hence, $q_L = \sum_{i=1}^{p} \frac{d_i\ (d_i-1)}{2}$

$$= \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - \frac{1}{2}\left(\sum_{i=1}^{p} d_i\right)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - \frac{1}{2}(2q)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - q$$

Definition: A graph G is called a ***line graph*** if G ≅ L(H) for some graph H.

Example: Consider the graph given in Fig. 2.2.

Clearly, $K_4 - x$ is a line graph.

MATRICES

ADJACENCY MATRIX

Definition: Let G = (V, X) be a (p, q) graph. Let V = {v_1, v_2, …, v_p}. The p x p matrix A = (a_{ij}) where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

is called the ***adjacency matrix***.

Example: The adjacency matrix of the graph G given in Fig. 1.35 is shown below:

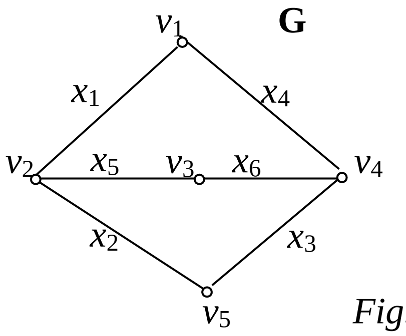


Fig. 1.35

Adjacency Matrix

	v_1	v_2	v_3	v_4	v_5
v_1	0	1	0	1	0
v_2	1	0	1	0	1
v_3	0	1	0	1	0
v_4	1	0	1	0	1
v_5	0	1	0	1	0

Remark:

1. The adjacency matrix A is symmetric.
2. The sum of the i^{th} row of A is equal to the degree of v_i.
3. The entries along the principal diagonal of A are 0.

INCIDENCE MATRIX

Definition: Let G = (V, X) be a (p, q) graph. Let V = $\{v_1, v_2, \ldots, v_p\}$ and

X = $\{x_1, x_2, \ldots, x_q\}$. The $p \times q$ matrix B = (a_{ij}) where

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent with } x_j \\ 0, & \text{otherwise} \end{cases}$$

is called the ***incidence matrix*** of the graph.

Example: The incidence matrix of the graph G given in Fig. 1.35 is shown below:

Incidence matrix

B =

	x_1	x_2	x_3	x_4	x_5	x_6
v_1	1	0	0	1	0	0
v_2	1	1	0	0	1	0
v_3	0	0	0	0	1	1
v_4	0	0	1	1	0	1
v_5	0	1	1	0	0	0

Remark:

1. The sum of the i^{th} row of B is equal to the degree of v_i.
2. Each column of the incident matrix B contains exactly two 1"s because each edge is incident with exactly two vertices.

Exercises:

1. Write the adjacency and incidence matrix for the graph G given in Fig.1.34.
2. Relabel the points of the graphs given in Fig. 1.30 and Fig. 1.32 and write the incidence and adjacency matrices for the relabeled graph.

OPERATIONS ON GRAPHS

Definition:

Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be two graphs with $V_1 \cap V_2 = \varphi$. Then

(i).The ***union*** $G_1 U\, G_2$ to be the graph (V, X) where $V = V_1 \, U \, V_2$ and

$X = X_1 \, U \, X_2$.

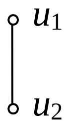
(ii). The ***sum*** $G_1 + G_2$ as $G_1 U\, G_2$ together with all the lines joining points of V_1 to points of V_2.

(iii). The ***product*** $G_1 \times G_2$ is the graph having vertex set $V = V_1 \times V_2$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$.

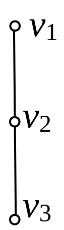
(iv). The composition $G_1[G_2]$ is the graph having vertex set $V_1 \times V_2$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if u_1 is adjacent to v_1 in G_1 or ($u_1 = v_1$ and u_2 is adjacent to v_2 in G_2).

Example:

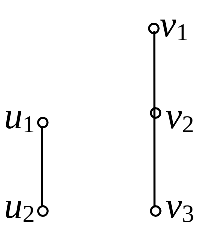
G_1



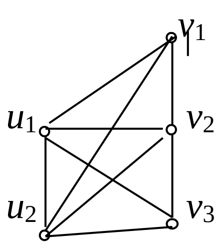
G_2



(G_1 U G_2)



$G_1 + G_2$



G_1 x G_2

G_1 [G_2]

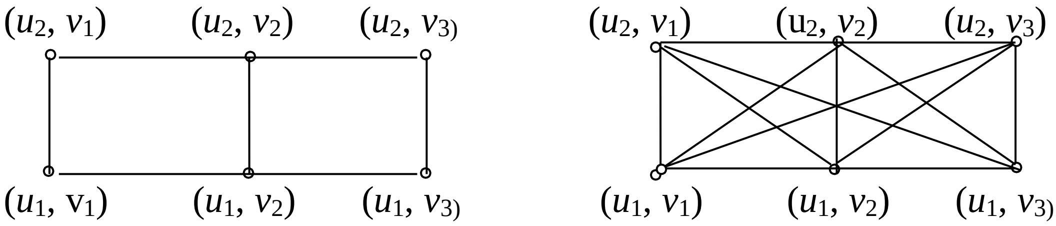


Fig. 2.3

Note: $\overline{K_m} + \overline{K_n} = K_{m,n}$.

Theorem 2.3: Let G_1 be a (p_1, q_1) graph and G_2 be a (p_2, q_2) graph.

(i). $G_1 U G_2$ is a $(p_1 + p_2, q_1 + q_2)$ graph.

(ii). $G_1 + G_2$ is a $(p_1 + p_2, q_1 + q_2 + p_1 p_2)$ graph.

(iii). $G_1 \times G_2$ is a $(p_1 p_2, q_1 p_2 + q_2 p_1)$ graph.

(iv). $G_1 [G_2]$ is a $(p_1 p_2, p_1 q_2 + p_2^2 q_1)$ graph.

Proof:

(i). Let G_1 be a (p_1, q_1) graph and G_2 be a (p_2, q_2) graph.

We know that, $G_1 U G_2$ is a graph with vertex set $V = V_1 U V_2$

and $X = X_1 U X_2$.

$\therefore |V_1 U V_2| = p_1 + p_2$ and $|X_1 U X_2| = q_1 + q_2$.

Hence $G_1 U G_2$ is a $(p_1 + p_2, q_1 + q_2)$ graph.

(ii). We know that, $G_1 + G_2$ is a graph with vertex set $V = V_1 U V_2$.

$\therefore |V_1 U V_2| = p_1 + p_2$ and

Number of lines in $G_1 + G_2$ = number of lines in $G_1 U G_2$ + number of lines joining points of V_1 to the points of V_2.

$= q_1 + q_2 + p_1 p_2.$

$\therefore$ $G_1 + G_2$ is a $(p_1 + p_2, \ q_1 + q_2 + p_1 p_2)$ graph.

(iii). We know that, $G_1 \times G_2$ is a graph with vertex set $V = V_1 \times V_2$.

$\therefore |V_1 \times V_2| = p_1 p_2.$

Also we know that, the points (u_1, u_2) and (v_1, v_2) are adjacent if $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$.

$\therefore$ deg (u_1, u_2) = deg u_2 + deg u_1

i.e. deg (u_1, u_2) = deg u_1 + deg u_2.

The total number of lines in $G_1 \times G_2$ $= \frac{1}{2}\left[\sum_{i=1}^{p1} \sum_{j=1}^{p_2} deg(u_i, u_j)\right]$

$$= \frac{1}{2}\left[\sum_{i=1}^{p1} \sum_{j=1}^{p2} (deg u_i + deg_j)\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (deg u_i\right] + \frac{1}{2}\left[\sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (deg u_j\right]$$

$$= \frac{1}{2}\left[p_2 2q_1 + p_1 2q_2\right]$$

$$= p_2 q_1 + p_1 q_2$$

Thus $G_1 \times G_2$ is a $(p_1 p_2, \ p_2 q_1 + p_1 q_2)$ graph.

(iv). $G_1[G_2]$ is the graph with vertex set $V_1 \times V_2$

$\therefore |V_1 \times V_2| = p_1 p_2.$

Also, we know that, the points (u_1, u_2) and (v_1, v_2) are adjacent in $G_1[G_2]$ if u_1 is adjacent to v_1 in G_1 or ($u_1 = v_1$ and u_2 is adjacent to v_2 in G_2).

$\therefore$ deg (u_1, u_2) = deg u_2 + p_2 deg u_1

$$= p_2 \deg u_1 + \deg u_2$$

The total number of lines in $G_1[G_2]$

$$= \frac{1}{2}\left[\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} deg(u_i, u_j)\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} (p_2\, deg u_i + deg_j)\right]$$

$$= \frac{1}{2}\left[p_2 \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} (deg u_i)\right] + \frac{1}{2}\left[\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} (deg u_j)\right]$$

$$= \frac{1}{2}\left[p_2^2 2q_1 + p_1 2q_2\right]$$

$$= p_2^2 q_1 + p_1 q_2$$

Hence $G_1 [G_2]$ is a $(p_1 p_2,\ p_1 q_2 + p_2^2 q_1)$ graph.

Exercises:

1. What is $K_m + K_n$?
2. Express $K_4 - x$ in terms of K_2 and $\bar{K}_2$.
3. Express the graph G in Fig. 1. 35 in terms of $\bar{K}_3$ and $\bar{K}_2$.
4. Express the graph L (G) in Fig. 1. 34 in terms of K_1 and K_3.

DEGREE SEQUENCES

Definition: A ***partition*** of a non – negative n is a finite set of non – negative integers $d_1, d_2, \ldots, d_p$ whose sum is n. This partition is denoted by $(d_1, d_2, \ldots, d_p)$.

For example, the integer 5 has the following partitions.

5 = (2, 2, 1) or (4, 1) or (3, 2) or (3, 1, 1) or (2, 1, 1) or (1, 1, 1, 1, 1).

Definition: Let G be a (p, q) graph. The partition of $2q$ as the sum of the degree of its points is called the ***partition*** or the ***degree sequence*** of the graph.

Example: Consider the graph $K_{1, 2}$ given in Fig. 2.4.

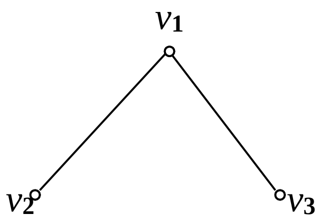


Fig. 2.4

Here, $d(v_1) = 2$, $d(v_2) = 1$, $d(v_3) = 1$

$\therefore$ degree sequence of $K_{1, 2} = (2, 1, 1)$.

GRAPHICAL PARTITION (OR) GRAPHIC SEQUENCE

Definition: A partition $P = (d_1, d_2, \ldots, d_p)$ of n into p parts is said to be a ***graphical partition*** or a ***graphic sequence*** if there exists a graph G whose points have degree d_i. G is called a ***realisation*** of P.

Example:

The partition P = (2, 1, 1) of 4 is graphical and $K_{1, 2}$ is the unique realization of P.

Remarks:

1. Any two isomorphic graphs determine the same partition. But the converse is not true.

   For example , the two *non – isomorphic* graphs given in Fig. 1.25 determine the same partition (3, 2, 1, 1, 1).

2. If the partition $(d_1, d_2, . . . , d_p)$ of n is graphical then n is even and $d_i \le p - 1$ for each i.

   This is the necessary condition that the sequence $(d_1, d_2, . . . , d_p)$ to be graphical . However the condition is not sufficient.

SOLVED PROBLEMS

Problem 1: Show that the partition P = (7, 6, 5, 4, 3, 2) is not graphic.

Solution:

Suppose P is graphic.

Then P has realization graph G.

Clearly, G has six points

Hence the maximum degree of any point in G is ≤ 5.

This is a contradiction to the degrees are 6 , 7.

$\therefore$ the given partition is not graphic.

Problem 2: Show that the partition P = (6, 6, 5, 4, 3, 3, 1) is not graphic.

Solution:

Suppose P is graphic.

Let G be its realization graph .

Clearly, G has seven points

Given that two points of G have degree 6.

$\therefore$ These two points are adjacent to every other point of G.

$\therefore$ The minimum degree of each vertex in G is at least 2.

This is a contradiction to the fact that a point has degree 1.

Hence P is not graphic.

Problem 3: Show that the partition P = (7, 6, 5, 4, 3, 2, 1) is not graphic.

Solution:

Suppose P is graphic.

Let G be its realization graph .

Clearly, G has seven points

Then G has seven points and maximum degree is 6.

This is a contradiction to the fact that the degree of a point is 7.

Hence P is not graphic.

GRAPHIC SEQUENCES

Theorem 2.4: [*The necessary and sufficient condition for a partition tobe graphical*]

A partition P = (d_1, d_2, . . . , d_p) of an even number into p parts with $p-1 \geq d_1 \geq d_2 \geq \ldots \geq d_p$ is graphical iff the modified partition P' =($d_2 -1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p$) is graphical.

Proof:

Assume that the modified partition

P' =($d_2 -1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p$) is graphical.

Let G' be its realisation graph with vertex set {v_2,v_3, . . . , v_p} such that

$d(v_2) = d_2 - 1$, $d(v_3) = d_3 - 1, \ldots, d(v_{d_1+1}) = d_{d_1+1}, \ldots, d(v_p) = d_p$.

Let G be a graph obtained from G' by adding a new vertex v_1 and making it adjacent to $v_2, v_3, \ldots, v_{d_1+1}$.

Clearly, the partition of G is $(d_1, d_2, \ldots, d_p)$.

Hence P is a graphic sequence.

Conversely, suppose P is graphical.

Let G = (V, X) be a realization graph of P with vertex set

$V = \{v_1, v_2, \ldots, v_p\}$ and $deg\ v_i = d_i$.

If v_1 is adjacent to $v_2, v_3, \ldots, v_{d_1+1}$ then $G' = G - \{v_1\}$ is a realization graph of P^1.

$\therefore P' = (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p)$

$\Rightarrow$ the modified partition P^1 is graphical.

If v_1 is not adjacent to all the vertices $v_2, v_3, \ldots, v_{d_1+1}$ then there exist two vertices v_i and v_j such that $d_i > d_j$ and v_1 is adjacent to v_j but not adjacent to v_i.

Since $v_1 v_i$ is not an edge, there exist a vertex v_k such that v_k is adjacent to v_i but not adjacent to v_j.

Let G' be the graph obtained from G by deleting the lines $v_1 v_j$ and $v_i v_k$ and by adding the lines $v_1 v_i$ and $v_j v_k$.

Clearly G' is a realisation of P in which v_1 is adjacent to v_I but not with v_j.

By repeating this process we get a realisation of P in which v_1 is adjacent to all the vertices $v_2, v_3, \ldots, v_{d_1+1}$.

Thus the modified partition P' is graphical.

Hence the theorem.

Note:

The above theorem gives an effective algorithm to determine whether a given partition P is graphical and to obtain a graph G realising P when it is graphical.

Algorithm:

Let $P = (d_1, d_2, \ldots, d_p)$ be a partition of an even integer with

$p - 1 \geq d_1 \geq d_2 \geq \ldots \geq d_p$. P is graphical iff the following procedure results in

a partition with every summand zero.

1). Determine the modified partition P^1 described in theorem 1. 12.

i.e. $P' = (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p)$

2). Reordering the terms of P' so that they are non – increasing and call the resulting partition P_1.

3). Determine the modified partition P'' of P_1 and let P_2 be the reordered partition.

4). Continue this process as long as non – negative summands can be obtained.

SOLVED PROBLEMS

Problem 1: Prove that the partition P = (6, 6, 5, 4, 3, 3, 1) is not graphical.

Solution:

Given partition is P = (6, 6, 5, 4, 3, 3, 1).

$= (d_1, d_2, d_3, d_4, d_5, d_6, d_7)$

$d_1 = 6$, $d_{d_1+1} = d_{6+1} = d_7$, $d_{d_1+1} - 1 = d_7 - 1 = 1 - 1 = 0$

The modified partition is

$P' = (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p)$.

$= (5, 4, 3, 2, 2, 0)$.

$P'' = (3, 2, 1, 1, -1)$.

It contains the negative number.

$\therefore P''$ is not graphical.

$\therefore$ P is not graphical.

Problem 2: Prove that the partition P = (4, 4, 4, 2, 2, 2) is graphical and construct graphs realizing the partition.

Solution: Let P = (4, 4, 4, 2, 2, 2) .

The modified partition is $P' = (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p)$.

$= (3, 3, 1, 1, 2)$, since $d_{d_1+1} - 1 = d_5 - 1 = 2 - 1 = 1$

$= (v_2, v_3, v_4, v_5, v_6)$, say

$P_1 = (3, 3, 2, 1, 1) = (v_2, v_3, v_6, v_4, v_5)$

$P_1' = (2, 1, 0, 1) = (v_3, v_6, v_4, v_5)$

$P_2 = (2, 1, 1, 0) = (v_3, v_6, v_5, v_4)$

Realisation of graph P_2:

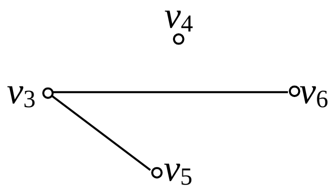


Fig. 2.5

Realisation of graph P_1:

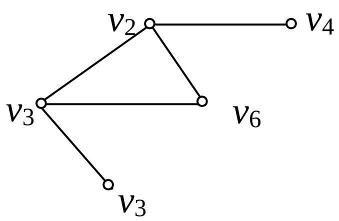


Fig. 2.6

Realisation of graph P:

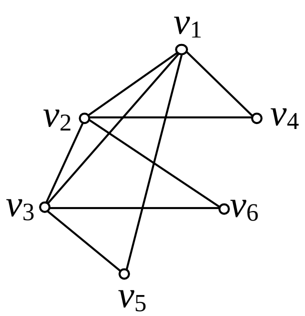


Fig. 2.7

P_2 is graphical

$\therefore$ P_1 is graphical

Hence, P is graphical.

Problem 3: Which of the following partitions are graphical? Wherever graphical, construct graphs realizing the partitions.

(a). (5, 5, 3, 3, 2, 2)

(b). (5, 3, 2, 1,1, 1,1,1,1)

(c). (7, 6, 5, 4, 3, 3, 2)

(d). (4, 3, 2, 1, 1, 1)

(e). (5, 3, 3,3, 3, 3)

Solution:

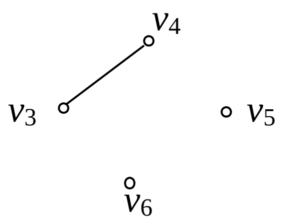
(a). Partition P = (5, 5, 3, 3, 2, 2) = ($v_1, v_2, v_3, v_4, v_5, v_6$)

Modified partitions

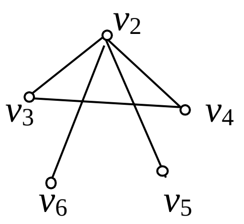
$P' = (4, 2, 2, 1, 1) = (v_2, v_3, v_4, v_5, v_6)$

$P'' = (1, 1, 0, 0) = (v_3, v_4, v_5, v_6)$

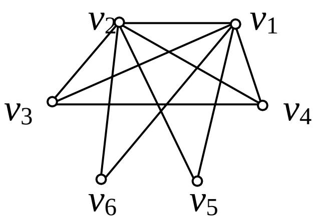
<u>Realisation of P''</u> :



<u>Realisation of P'</u> :



<u>Realisation of</u> :



P'' is graphical.

$\therefore$ P' is graphical.

Hence, P is graphical.

(b). Let P $= (5, 3, 2, 1,1, 1, 1, 1, 1) = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)$

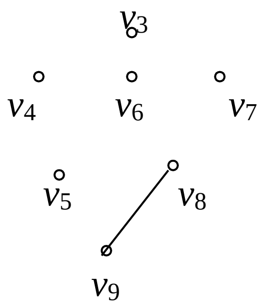
$P' = (2, 1, 0, 0, 0, 1, 1, 1) = (v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)$

$P_1 = (2, 1, 1, 1, 1, 0, 0, 0) = (v_2, v_3, v_7, v_8, v_9, v_4, v_5, v_6)$

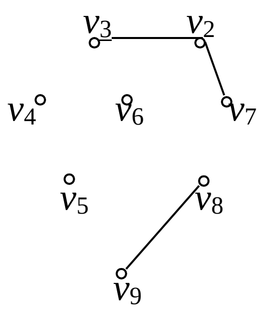
$P_1' = (0, 0, 1, 1, 0, 0, 0) = (v_3, v_7, v_8, v_9, v_4, v_5, v_6)$

$P_2 = (1, 1, 0, 0, 0, 0, 0, 0) = (v_8, v_9, v_3, v_7, v_4, v_5, v_6)$

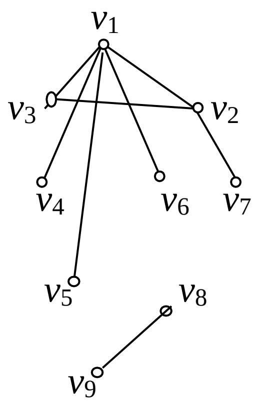
Realisation of P_2:



Realisation of P_1



Realisation of P:



P_2 is graphical

$\therefore$ P_1 is graphical.

Hence, P is graphical.

(c) Let P = (7, 6, 5, 4, 3, 3, 2)

Suppose P is graphical

Let G be its realization graph. Then G contains 7 points and the maximum degree is 6.

This is a contradiction, since here the degree is 7.

∴ P is not graphical.

Theorem 2.6: If a partition P = ($d_1, d_2, \ldots, d_p$) with $d_1 \geq d_2 \geq \ldots \geq d_p$ is graphical then $\sum_{i=1}^{p} d_i$ is even and $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{p} min\{k, d_i\}$ for $1 \leq k \leq p$.

Proof:

Given that the partition P = ($d_1, d_2, \ldots, d_p$) is graphical.

Let G = (V , X) be the realization of P with vertex set V = {$v_1, v_2, \ldots, v_p$}

and $deg\ v_i = d_i$.

We know that, by theorem 1.1

$$\sum_{i=1}^{p} d_i = 2q = \text{even number.}$$

$$\therefore \sum_{i=1}^{p} d_i \text{ is even.}$$

Now the sum $\sum_{i=1}^{k} d_i$ is the sum of the degrees of the vertices $v_1, v_2, \ldots, v_k$.

This can be divided into two parts.

The first part contains the lines joining the points $v_1, v_2, \ldots, v_k$.

This part is $\leq k(k-1)$.

The second part contains lines joining one of the points

$\{ v_{k+1}, v_{k+2}, \ldots, v_p \}$ with the points on the set $\{ v_1, v_2, \ldots, v_k \}$.

Clearly the second part is $\leq \sum_{i=k+1}^{p} mi\{k, d_i\}$

Hence, $\sum_{i=1}^{p} d_i \leq k(k-1) + \sum_{i=k+1}^{p} min\{k, d_i\}$.

UNIT – III

Walks - trails and Paths - connectedness and components - blocks - connectivity

WALKS, TRIALS AND PATHS

WALK

Definition: A ***walk*** of a graph G is defined as a finite alternating sequence of points and lines of the form $v_0, x_1, v_1, x_2, v_2, x_2, v_3, \ldots, v_{n-1}, x_n, v_n$ beginning and ending with points such that each line x_i is incident with v_{i-1} and v_i.

Definition: The walk joins v_0 and v_n is called a $\mathbf{v_0}$ **-** $\mathbf{v_n}$ walk. v_0 is called the ***initial point*** and v_n is called the ***terminal point*** of the walk. The number of lines in the walk is called the ***length of the walk***.

Note:

1). No edge appears more than once in a walk.

2). A vertex may appear more than once.

3). The v_0 - v_n walk is also denoted by $v_0, v_1, \ldots, v_n$.

4). A single point is considered as a walk of length 0.

CLOSED WALK AND OPEN WALK

Definition: A walk which begins and ends at the same point is called a ***closed walk***

i.e. a $v_0 - v_n$ walk is called walk if $v_0 = v_n$.

A walk that is not closed is called an ***open walk***.

CYCLE

Definition: A closed walk in which no point except the terminal point appear more than once is called a ***cycle***.

A closed walk $v_0, v_1, v_2, \ldots, v_n = v_0$ in which $n \geq 3$ and $v_0, v_1, v_2, \ldots, v_{n-1}$ are distinct is called a *cycle of length n*.

The graph consisting of cycle of length n is denoted by C_n.

C_3 is called a triangle.

TRIAL

Definition: A walk is called a ***trial*** if all its lines are distinct.

PATH

Definition: A walk is called a ***path*** if all its points are distinct.

Note:

1).Every path is a trial and a trail need not be a path.

2). The graph consisting of a trial with *n* points is denoted by P_n .

3).The *length of a path* in which the number of lines in the path.

Example: Consider the graph given in Fig. 3.1.

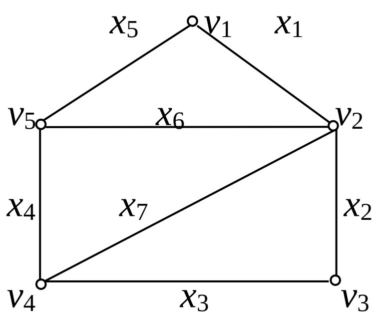


Fig. 3.1

1). v_1, v_2, v_3, v_4, v_5 is a walk. It is a $v_1 - v_5$ walk.

Initial point of the walk is v_1. Terminal point of the walk is v_5.

The length of the walk is 4.

2). $v_1, v_2, v_3, v_4, v_2, v_1, v_2, v_5$, is a walk.

3). v_1, v_2, v_4, v_5, v_2 and $v_1, v_2, v_3, v_4, v_5, v_1$ are closed walk.

4). $v_1, v_2, v_4, v_3, v_2, v_5$ is a trial but not a path.

5). v_1, v_2, v_4, v_5 is a path.

6). v_2, v_3, v_4, v_2 is a cycle.

Theorem 3.1: In a graph G, any $u - v$ walk contains a $u - v$ path.

Proof: We prove the result by induction on the length of the walk.

Any walk of length zero or one itself is a path.

Assume the result for all walks of length $< n$.

Prove the result for a walk of length n.

Let $u = u_0, u_1, u_2, \ldots, u_n = v$ be a $u - v$ walk of length n.

If all the points $u_0, u_1, u_2, \ldots, u_{n\ a}$ are distinct then this walk itself is a path.

If not, there exist i and j such that $0 \leq i < j \leq n$ and $u_i = u_j$.

Now $u = u_0, u_1, u_2, \ldots, u_i, u_{j+1}, \ldots, u_n = v$ is a $u - v$ walk of length $< n$.

$\therefore$ By induction hypothesis this walk contains $u - v$ path.

Hence, any $u - v$ walk contains a $u - v$ path.

Theorem 3.2: If $\delta \geq k$ then G is a path of length k.

Proof:

Let δ be the minimum degree of the graph G.

Let k be the number of vertices of the graph G.

Let P = $\{v_0, v_1, v_2, \ldots, v_n\}$ be the longest path in G.

Then every vertex adjacent to v_0 lies on P.

Since $deg\, v_0 \geq \delta$, the length of P $\geq \delta$ and $\delta \geq k$.

Hence $P_1 = \{ v_0, v_1, v_2, \ldots, v_k\}$ is a path of length k in G.

Theorem 3.3: A closed walk of odd length contains a cycle.

Proof:

Let $v_0, v_1, v_2, \ldots, v_n = v_0$ be a closed walk of odd length n .

Clearly $n \geq 3$.

If $n = 3$ then the closed walk of length three is a triangle which is trivially a cycle.

$\therefore$ The result is true for n = 3 .

Assume the result is true for all walks of length $< n$.

To prove the result for a closed walk $v_0, v_1, v_2, \ldots, v_n = v_0$ of odd length n .

If all the points in this walk are distinct then this walk itself is a cycle.

If not there exist two positive integers i and j such that i < j ,

$\{i , j\} \neq \{ 0 , n\}$ and $v_i = v_j$, where n is odd.

Now $v_i, v_{i+1}, \ldots, v_j$ and $v_0, v_1, v_2, \ldots, v_i, v_{j+1}, \ldots v_n = v_0$ are closed walks contained in the given walk and the sum of their lengths is n.

Since n is odd, at least one of these walks is of odd length.

Hence by induction hypothesis this closed walk contains a cycle.

$\therefore$ By the principle of Mathematical induction, the theorem is true for all odd length n.

SOLVED PROBLEMS

Problem 1: If A is the adjacency matrix of a graph with $V = \{v_1, v_2, \ldots, v_p\}$ then prove that for $n \geq 1$ the $(i, j)^{th}$ entry of A^n is the number of $v_i - v_j$ walks of length n.

Solution:

Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_p\}$.

We know that ,

The adjacency matrix $A = (a_{ij})_{p \times p}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

We prove the result by induction on n.

When $n = 1$,

The number of $v_i - v_j$ walks of length 1.

$$= \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

$= (a_{ij}) = (i\,,j)^{th}$ element of A.

$\therefore$ The result is true for $n = 1$.

Assume that the result is true for $n - 1$.

Let $A^{n-1} = (a_{ij}^{(n-1)})$

$\therefore\ {}^{(n-1)}_{ij}$ = the number of $v_i - v_j$ walks of length $(n - 1)$.

Now $A^n = A^{n-1}.\,A$

$= ({}^{(n-1)}_{ij})_{p \times p}\,(a_{ij})_{p \times p}$

$\therefore (i\,,j)^{th}$ entry of $A^n = \sum_{i=1}^{p} {}^{(n-1)}_{ik}\, a_{kj}$ ……….(1).

Take $${}^{(n-1)}_{ik}\, a_{kj} = \begin{cases} {}^{(n-1)}_{ij}, & \text{if } a_{kj} = 1 \text{ i.e. if } v_k \text{ is adjacent to } v_j \\ 0, & \text{if } v_k \text{ is not adjacent to } v_j \end{cases}$$

By induction hypothesis the $(i\,,j)^{th}$ entry of A^{n-1} is the number of walks of length n - 1 between v_i and v_k if v_k is adjacent to v_j then the above walk can be made into walks of length n between v_i and v_j.

$\therefore (i\,,j)^{th}$ entry of A^n is the number of walks of length n between v_i and v_j.

Hence the theorem.

CONNECTEDNESS AND COMPONENTS

CONNECTEDNESS

Definition: ***Two points*** u and v of a graph G are said to be ***connected*** if there exists a $u - v$ path.

Definition: A *graph* G is said to be ***connected*** if there is at least one path between every pair of vertices in G.

A graph G which is not connected is said to be ***disconnected***.

COMPONENTS

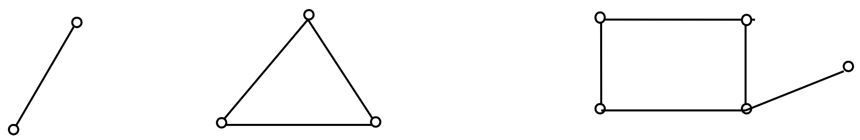
Definition: Each of the connected graphs is called a ***component***.

A graph G is connected iff it has exactly one component.

A graph G is disconnected then G has at least two components.

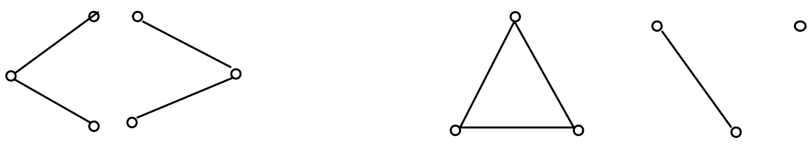
Example:

Connected Graphs



G_1 **G_2** **G_3**

Disconnected Graphs



G_4 *Fig.* 3.2 **G_5**

G_1 is a connected graph with one component.

G_2 is a connected graph with one component.

G_3 is a connected graph with one component.

G_4 is a disconnected graph with two components.

G_5 is a disconnected graph with three components.

Theorem 3.4: A graph G with p points and $\delta \geq \frac{p-1}{2}$ is connected.

Proof:

Let G be a graph and $\delta \geq \frac{p-1}{2}$ is connected.

To prove: G is connected.

Suppose G is not connected.

Then G has at least two components.

Consider $G_1 = (V_1, X_1)$ of G.

Let $v_1 \in V_1$.

We have $\delta \geq \frac{p-1}{2}$.

$\therefore \exists$ at least $\frac{p-1}{2}$ points in G_1 which are adjacent to V_1.

$\therefore$ V_1 contains at least $\frac{p-1}{2} + 1$ points.

i.e. V_1 contains $\frac{p+1}{2}$.

Also G has at least two components.

$\therefore$ The number of points in G $\geq \frac{p+1}{2} + \frac{p+1}{2}$

i.e. $p \geq p + 1$

Which is a contradiction to G has p points.

Hence G is connected.

Theorem 3.5: A graph G is connected iff for any partition of V into subsets V_1 and V_2 there is a line of G joining a points of V_1 to a point of V_2.

Proof:

Assume that G is connected.

Let $V = V_1 U\ V_2$ and $V_1 \cap V_2 = \varphi$.

To prove: There is a line of G joining a point of V_1 to a point of V_2.

Let $u \in V_1$ and $v \in V_2$.

Since G is connected, there exists a u - v path in G.

Let $u = v_0, v_1, v_2, \ldots, v_n = v$ be a path.

Let i be the least positive integer such that $v_i \in V_2$.

Then $v_{i-1} \in V_1$.

$\therefore$ The line $v_{i-1} v_i$ joins a point of V_1 to a point of V_2.

Hence there is a line of G joining a point of V_1 to a point of V_2.

Conversely, assume that there is a line of G joining a point of V_1 to a point of V_2.

To prove: G is connected.

Suppose G is not connected.

Then G contains at least two components say, G_1 and G_2.

Let V_1 be the set of all points of G_1 and V_2 be the set of all points of V_2.

Clearly $V = V_1 U\ V_2$ is a partition of V.

Also there is no line joining any point of V_1 to a point of V_2.

Which is a contradiction to the assumption.

$\therefore$ G is connected.

Theorem 3.6: If G is not connected then $\bar{G}$ is connected.

Proof:

Suppose G is not connected.

Then G has at least two components.

Let *u* and *v* be any two points of G.

If *u* and *v* belong to different components of G then they are not adjacent in G.

$\therefore$ They are adjacent in $\bar{G}$. Hence $\bar{G}$ is connected.

If *u* and *v* lie in the same component of G then they are connected in G.

Choose w in the other component.

Then *u*, *w*, *v* is a *u* – *v* path in $\bar{G}$.

Hence $\bar{G}$ is connected.

DISTANCE

Definition: For any two points *u*, *v* of a graph we define the distance between *u* and *v* by

$$d(u, v) = \begin{cases} \text{the length of a shortest } u - v \text{ path if such a path exists} \\ \infty \ , \quad \text{otherwise} \end{cases}$$

Note:

If G is a connected graph then $d(u, v)$ is always a non negative integer.

Hence, d is a metric on the set of points V.

Theorem 3.7: [*Necessary and sufficient condition for a graph to be bipartite***]**

A graph G with at least two points is bipartite iff all its cycles are of even length.

Proof:

Let G be a graph with at least two points is bipartite.

Then V can be partitioned into two subsets V_1 and V_2 such that every line joins a point of V_1 to a point of V_2.

To prove: All its cycles are of even length.

Let $v_0, v_1, v_2, \ldots, v_n = v_0$ be a cycle of length n .

Choose $v_0 \in V_1$.

Then $v_2, v_4, v_6, \ldots \in V_1$ and $v_1, v_3, v_5, \ldots \in V_2$.

Also $v_n = v_0 \in V_1$.

$\therefore$ n is even.

Hence all its cycles are of even length.

Conversely, assume that all cycles in g are of even length.

Without loss of generality we assume that G is connected.

To prove: G is bipartite.

Let $v_1 \in V_1$.

Define $V_1 = \{ v \in V / d(v, v_1) \text{ is even} \}$

$V_2 = \{ v \in V / d(v, v_1) \text{ is odd} \}$

Clearly, $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \varphi$.

Claim: Every line of G joins a point of V_1 to a point of V_2.

Suppose two points $u, v \in V_1$ are adjacent.

Let P be the shortest $v_1 - u$ path of length m and let Q be the shortest $v_1 - v$ path of length n.

Since $u, v \in V_1$, both m and n are even.

Let u_1 be the last common point of P and Q.

Then the $v_1 - u_1$ path along P and the $v_1 - u_1$ path along Q are both shortest path and hence have the same length, say i.

Now the $u_1 - u$ path along P, the line $u\,v$ followed by the $v_1 - u_1$ path along Q form a cycle.

Its length is $= (m - i) + 1 + (n - i) = m + n - 2i + 1 =$ odd number.

This is a contradiction to our assumption.

Hence no two points of V_1 are adjacent.

Similarly, we can prove that no two points of V_2 are adjacent.

Thus every line joins a point of V_1 to a point of V_2.

Hence G is bipartite.

CUT POINT

Definition: A ***cut point*** of a graph G is a point whose removal increases the number of components.

BRIDGE

Definition: A ***bridge*** of a graph G is a line whose removal increases the number of components.

Note: If v is a cut point of a connected graph then G – {v} is disconnected.

Example:

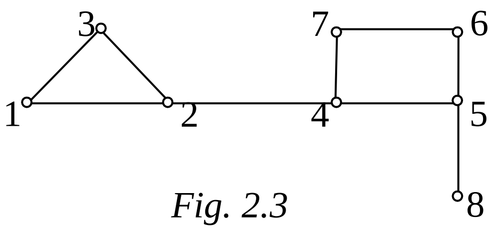


Fig. 2.3

For the graph given in Fig. 2.3,

2, 4, 5 are cut points.

{2, 4} and {5, 8} are bridges.

Theorem 3.8: Let v be a point of a connected graph G. Then the following statements are equivalent.

(i). v is a cut - point of G.

(ii). There exists a partition of V – {v} into subsets U and W such that for each $u \in$ U and $w \in$ W, the point v is on every $u - w$ path.

(iii). There exists two points u and w distinct from v such that v is on every $u - w$ path.

Proof: Let G be a connected graph and v be a point of G.

To prove: (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

First, we prove: (i) ⇒ (ii).

Assume that v is a cut - point of G.

Then G – v is disconnected.

$\therefore$ G – v contains at least two components.

Let U be the set of points in one component and W be the set of points in the remaining components.

$\therefore$ V – {v} = U ∪ W and U ∩ W = φ .

i.e. There exists a partition of V – {v} into subsets U and W.

To prove: each $u \in$ U and $w \in$ W, the point v is on every $u - w$ path.

Let $u \in$ U and $w \in$ W.

Then u and w lie on different component of G – v.

$\therefore$ There is no $u - w$ path in G – v.

Hence the point v lies on every $u - w$ path in G.

Secondly, to prove: (ii) ⇒ (iii)

This is trivially true.

Thirdly, to prove: (iii) ⇒ (i)

Assume that there exists two points u and w distinct from v such that v is on every $u - w$ path.

$\therefore$ there is no $u - w$ path in G – v.

⇒ G – v is disconnected.

Hence v is a cut - point of G.

Theorem 3.9: Let x be a line of a connected graph G. Then the following statements are equivalent.

(i). v is a bridge of G.

(ii).There exists a partition of V into subsets U and W such that for

every point $u \in$ U and $w \in$ W, the line x is on every $u - w$ path.

(iii).There exists two points u and w such that the line x is on every

$u - w$ path.

The Proof is similar to theorem 3.8 and is left as an exercise.

Theorem 3.10: A line x of a connected graph G is a bridge iff x is not on any cycle of G.

Proof: Let G be a connected graph.

Let the line x be a bridge of G ....(1).

Then G – x is disconnected.

To prove: x is not on any cycle C of G.

Suppose x is on any cycle C of G.

Let w_1 and w_2 be any two points in G.

Since G is connected, there exists a $w_1 - w_2$ path P in G.

If x is not on P, then P itself is a $w_1 - w_2$ path in G – x.

$\therefore$ G – x is connected , which is a contradiction to (1).

If x is on P, then replace x by C – x , we get a $w_1 - w_2$ walk in G – x.

This walk contains a $w_1 - w_2$ path in G – x.

$\therefore$ G – x is connected , which is a contradiction to (1).

Conversely, assume that x is not on any cycle of G ....(2)

To prove: x is a bridge of G.

Suppose x is not a bridge.

Then G – x is connected.

Let $x = u\,v$.

Then there exists a u - v path in G – x.

This path together with the line $x = u\,v$ forms a cycle containing x .

This is a contradiction to (2). i.e. to x is not on any cycle of G.

Hence x is a bridge.

Theorem 3.11: Every non – trivial connected graphs has at least two points which are not cut points.

Proof: Let G be a non – trivial connected graph.

Choose two points u and v such that $d\,(u\,,\,v)$ is maximum.

To prove: u and v are not cut points.

Suppose v is a cut point.

Then G – v is disconnected.

$\therefore$ G – v has at least two components.

Choose a point w in a component that do not contain u.

Then v lies on every u – w path.

$\therefore\; d\,(u\,,\,w) > d\,(u\,,\,v)$.

This is a contradiction to $d\,(u, v)$ is maximum.

$\therefore v$ is not a cut point.

Similarly we can prove that u is not a cut point.

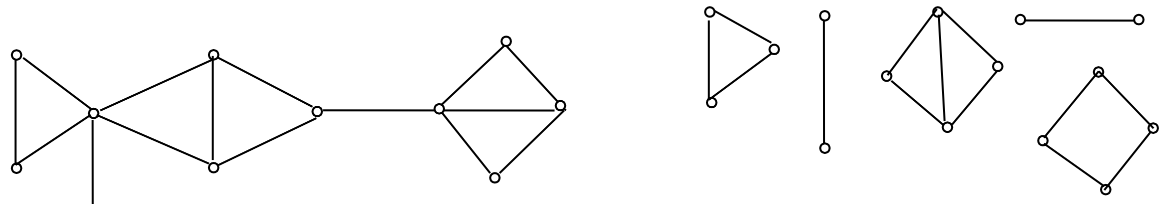
Hence G has at least two points which are not cut points.

BLOCKS

Definition: A connected non – trivial graph having no cut point is a ***block***.

A block of a graph is a sub graph and which is connected.

Example: A graph and its blocks are given in Fig. 3.4.



G ***Blocks of G***

***Fig.*3.4**

Theorem 3.12: Let G be a connected graph with at least three points. The following statements are equivalent.

(1). G is a block.

(2). Any two points of G lie on a common cycle.

(3). Any point and any line of G lie on a common cycle.

(4). Any two lines of G lie on a common cycle.

Proof: Let G be a connected graph with at least three points.

(i). To prove: (1) ⇒ (2).

Assume that G is a block..

Then G has no cut points.

To prove that any two points of G lie on a common cycle.

Let u and v be any two points in G.

We prove the result by using induction on $d(u, v)$.

If $d(u, v) = 1$ then u and v are adjacent and $G \neq K_2$, Since G has at least three points.

Also G has no cut points.

$\therefore x = uv$ is not a bridge.

By theorem 2. 10, x lies on a common cycle.

Hence the points u and v lie on a common cycle of G.

Assume the result for any two points at distance less than k .

To prove the result for $d(u, v) = k$, where $k \geq 2$.

Consider a $u - v$ path of length k .

Let w be a point that precedes v on this path.

Then $d(u, w) = k - 1$.

By induction hypothesis, the points u and w lie on a common cycle C of G.

Since G is a block, w is not a cut point of G.

$\therefore$ G - w is connected.

Hence there exists a $u - v$ path not containing w .

Let v' be the last point common to P and C. [See Fig . 2. 5].

Since u is common to P and C , such a v' exists.

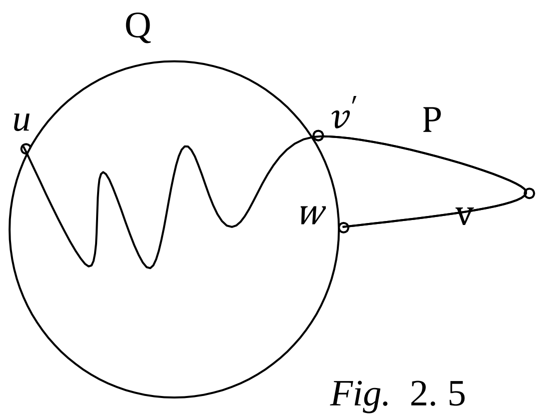


Fig. 2. 5

Let Q denote the $u - v'$ path along the cycle not containing the point w.

Now, $u - v'$ path along Q, v' - v path along P, the line $v\,w$ and $w - u$ path along C form a cycle.

This cycle contains both u and v.

Hence by induction, any two points of G lie on a common cycle.

(ii). To prove : (2) ⇒ (1).

Assume that any two points of G lie on a common cycle.

To prove that G is a block.

Since G is a connected non – trivial graph it is enough to prove that G has no cut point.

Suppose v is a cut point.

By theorem 4.8, there exists two points u and w distinct from v such that v lies on every $u - w$ path.

Also by assumption, u and w lie on a common cycle. This cycle determines two $u - w$ paths and at least one of these paths does not contain v .

This is a contradiction, since v lies on every $u - w$ path.

$\therefore$ v is a cut point.

Hence G is a block.

$\therefore$ (1) $\Longleftrightarrow$ (2).

(iii). To prove : (2) $\Rightarrow$ (3).

Assume that any two points of G lie on a common cycle of G.

To prove that any point and any line lie on a common cycle.

Let u be a point and $v\,w$ be a line of G.

By assumption u and v lie on a common cycle C.

If w lies on C, then the point u and the line $v\,w$ lie on a common cycle.

If w is not on C, let C' be a cycle containing u and w.

This cycle determines two $w - u$ paths and at least one of them does not contain v. Denote this path by P.

Let u' be the first point common to P and C.

Now, the line $v\,w$, the w - u' path along P, u' - v path in C containing u form a cycle. This cycle contains the point u and the line $v\,w$.

(iv). To prove : (3) $\Rightarrow$ (2) is trivial.

$\therefore$ (2) $\Longleftrightarrow$ (3).

(v). To prove : (3) $\Rightarrow$ (4) is trivial.

Assume that any point and any line lie on a common cycle.

To prove that any two lines of G lie on a common cycle.

Let $u\,u_1$ and $v\,w$ be two lines.

By assumption, the point u and the line $v\,w$ lie on a common cycle C.

Also the point u_1 and the line $v\,w$ lie on the common cycle C'.

Now the line $u\,u_1$, $u_1\,w$ path along C' , the line $v\,w$ and the $v - w$ path along C form a cycle.

This cycle contains the lines $u\,u_1$ and $v\,w$.

Hence any two lines of G lie on a common cycle.

(vi). To prove : (4) $\Rightarrow$ (3) is trivial.

$\therefore$ (3) $\Longleftrightarrow$ (4).

Hence, the statements (1) , (2) , (3) and (4) are equivalent for any connected graph with at least three points.

CONNECTIVITY

Definition: The ***connectivity*** $\kappa = \kappa$ (G) of a graph G is the minimum number of points whose removal gives a disconnected or trivial graph.

The ***line connectivity*** $\lambda = \lambda$(G) of a graph G is the minimum number of lines whose removal gives a disconnected or trivial graph.

Examples:

1). The connectivity of a disconnected graph is 0.

2). The line connectivity of a disconnected graph is also 0.

3). The connectivity of a connected graph with one cut point is 1.

4). The line connectivity of a connected graph with a bridge is 1.

5). For the complete graph K_p, $\kappa = p - 1 = \lambda$.

Theorem 3.13: For any graph G, $\kappa \leq \lambda \leq \delta$.

Proof:

First we prove that $\lambda \leq \delta$.

If G has no lines then $\lambda = 0$, $\delta = 0$.

Otherwise removal of all the lines incident with a vertex of minimum degree gives a disconnected graph.

$\therefore \lambda \leq \delta$ ....(1).

Now to prove $\kappa \leq \lambda$.

Case (*i*): G is disconnected or trivial.

Then $\lambda = 0$, $\delta = 0$.

Case (*ii*): G is a connected graph with a bridge $x = uv$.

Then $\lambda = 1$.

In this case $G = K_2$ or one of the points incident with x is a cut point.

$\therefore \kappa = 1.$

Hence $\lambda = \kappa = 1.$

Case (*iii*): $\lambda \geq 2.$

Then there exist λ lines whose removal gives a disconnected graph.

$\therefore$ the removal of $\lambda - 1$ lines gives a connected graph G with a bridge $x = uv$.

For each of these $\lambda - 1$ lines, elect an incident point different from u or v.

The removal of these $\lambda - 1$ points removes all the $\lambda - 1$ lines.

Hence the resulting graph is disconnected with a bridge $x = uv$.

$\therefore \kappa \leq \lambda - 1.$

Thus the removal of u or v gives a disconnected or trivial graph.

$\therefore \kappa \leq \lambda$**.**

Hence $\kappa \leq \lambda \leq \delta$.

Note: The inequality $\kappa \leq \lambda \leq \delta$ is often strict inequality. i.e. $\kappa < \lambda < \delta$.

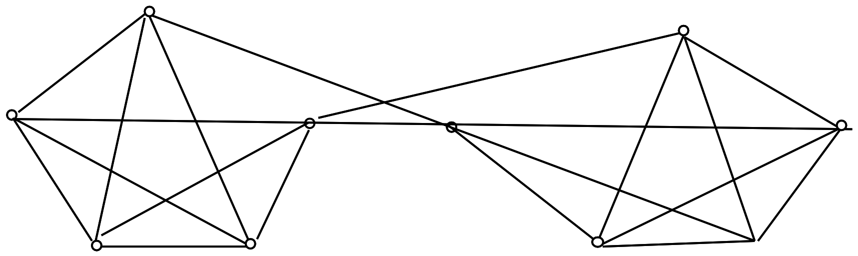


Fig. 3. 6

$\kappa = 2$, $\lambda = 3$ and $\delta = 4$.

Definition: A graph G is said to be n – ***connected*** if κ (G) $\geq n$ and n – ***line connected*** if λ (G) $\geq n$.

Note:

1). A non trivial graph is 1 – connected iff it is connected.

2). A non trivial graph is 2 – connected iff it is a block having more than one line. Hence K_2 is the only block which is not 2 – connected.

SOLVED PROBLEMS

Problem 1: Prove that if G is a k – connected graph then $q \geq \frac{pk}{2}$.

Solution: Let G be a (p , q) graph.

Since G is k – connected , $\kappa \geq k$.

$\therefore$ $k \leq \delta$, [since $\kappa \leq \lambda \leq \delta$]

Now $q = \frac{1}{2}\sum_{i=1}^{p} deg v_i$

$\geq \frac{1}{2} p\,\delta$, [since $deg\, v_i \geq \delta$]

$\geq \frac{pk}{2}$, [since $\delta \geq \kappa \geq k$]

Problem 2: Prove that there is no 3 – connected graph with 7 edges.

Solution: Suppose G is a 3 – connected graph with 7 edges.

Then $p \geq 5$ and $\kappa > 3$.

We have $q \geq \frac{3p}{2}$, [since if G is a k – connected graph then $q \geq \frac{pk}{2}$]

$\Rightarrow q \geq$ (3x5 / 2)

$\Rightarrow q \geq 7.5$

$\Rightarrow q \geq 8$, which is a contradiction since G has only 7 edges.

Hence there is no 3 – connected graph with 7 edges.

Problem 3: Find the connectivity of $K_{m,n}$.

Solution:

$$\text{Connectivity } \kappa = \min\{m, n\}$$

$$\lambda = \min\{m, n\}$$

$$\delta = \min\{m, n\}$$

$$\therefore \kappa = \lambda = \delta.$$

Unit IV

Eulerian graph and Hamiltonian graph

EULERIAN GRAPHS

Definition : A closed trial containing all points and lines is called an ***Eulerian trial .***

A graph having an Eulerian trial is called an ***Eulerian graph***.

Example:

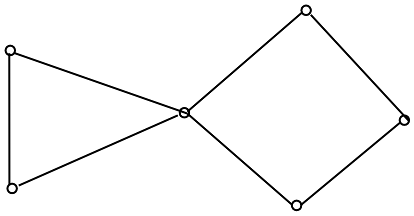


Fig. 4.1

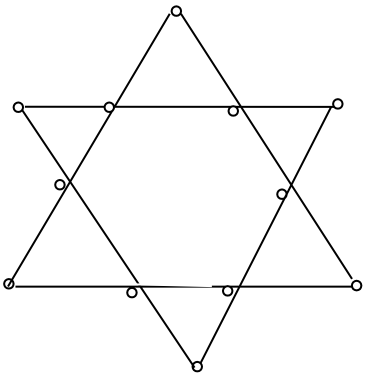


Fig. 4.2

The graph given in Fig. 4.1 and Fig. 4.1 are an Eulerian graphs.

Theorem 4.1: If G is a graph in which the degree of every vertex is atleast two then G contains a cycle.

Proof:

Construct a sequence $v, v_1, v_2, \ldots$ of vertices as follows.

Choose a vertex v.

Let v_1 be any vertex adjacent to v.

Let v_2 be any vertex adjacent to v_1 other than v.

At any stage, if vertex v_i , i $\geq$ 2 is already chosen, then choose v_{i+1} to be any vertex adjacent to v_i other than v_{i-1} .

Since degree of each vertex is at least 2, the existence of v_{i+1} is always guaranteed.

Since G has only a finite number of vertices, at some stage we have to choose a vertex which has been chosen before.

Let v_k be the first such vertex and let $v_k = v_i$ where $i < k$.

Then $v_i , v_{i+1}, \ldots, v_k$ is a cycle.

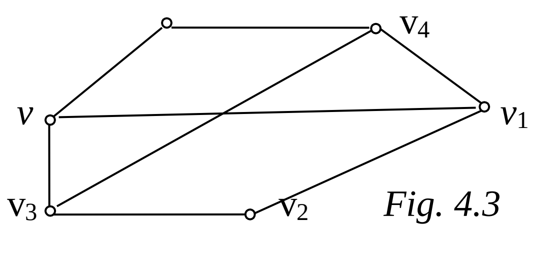


Fig. 4.3

Euler's problem:

In what type of graph G is it possible to find a closed trial running through every edge of G'?

Theorem 4.2: The following statements are equivalent for a connected graph G.

(1). G is Eulerian.

(2). Every point of G has even degree.

(3). The set of edges of G can be partitioned into cycles.

Proof:

(i). To prove (1) ⇒ (2):

Let G be an Eulerian graph.

Let T be an Eulerian trial in G with origin (and terminus) u .

Each time a vertex v occurs in T in a place other than the origin and terminus , two of the edges incident with v are accounted for.

Since an Eulerian trial contains every edges of G, $d(v)$ is even for every $v \neq u$.

For u , one of the edges incident with u is accounted for by the origin of T, another by the terminus of T and others are accounted for in pairs.

Hence $d(u)$ is also even.

(ii). To prove (2) ⇒ (3).

Since G is connected and non - trivial every vertex of G has degree at least 2, G contains a cycle Z.

The removal of the lines of Z results in a spanning sub graph G_1 in which again every vertex has even degree.

If G_1 has no edges then all the lines of G form one cycle and hence (3) holds.

Otherwise, G_1 has a cycle Z_1.

Removal of the lines of Z_1 from G_1 results in a spanning sub graph G_2 in which every vertex has even degree.

Continuing the above process, when a graph G_n with no edge is obtained, we obtain a partition of the edges of G into n cycles.

(iii). To prove (3) $\Rightarrow$ (1):

If the partition has only one cycle, then G is obviously Eulerian, since it is connected.

Otherwise let $Z_1, Z_2, \ldots, Z_n$ be the cycles forming a partition of the lines of G.

Since G is connected there exists a cycle $Z_i \neq Z_1$ having a common point v_1 with Z_1.

Without loss of generality, let it be Z_2.

The walk beginning at v_1 and consisting of the cycles Z_1 and Z_2 in succession is a closed trial containing the edges of these two cycles.

Continuing this process, we can construct a closed trial containing all the edges of G. Hence, G is Eulerian.

Corollary 1: Let G be a connected graph with exactly $2n$, $n \geq 1$, odd vertices. Then the edge set of G can be partitioned into n open trials.

Proof:

Let G be a connected graph with exactly $2n$, $n \geq 1$, odd vertices.

Let the odd vertices of G be labeled $v_1, v_2, \ldots, v_n$, $w_1, w_2, \ldots, w_n$ in any order.

$\deg(v_i)$ = odd number and $\deg(w_i)$ = odd number.

Add n edges $(v_1, w_1), (v_2, w_2), \ldots, (v_n, w_n)$ to G.

The resulting graph G' may be a multi graph.

No two of these n – edges are incident with the same vertex.

Also every vertex of G' is of even degree.

$\therefore G'$ has an Eulerian Trial T.

If we remove the n edges that we added to G from T then the open trial T will split into n open trials.

Hence the edge set of G can be partitioned into n – open trials.

Corollary 2: Let G be a connected graph with exactly two odd vertices. Then G has an open trial containing all the vertices and edges of G.

Proof:

This is only a particular case of corollary 1.

Obviously the open trial mentioned in corollary 2 begins at one of the odd vertices and end at the other.

Note: Corollary 2 answers the following question:

" Which diagram can be drawn without lifting one‟s pen from the paper not covering any line segment more than once ? ".

TRACEABLE (OR) ARBITRARILY TRAVERSABLE

Definition: A graph G is said to be ***arbitrarily traversable or traceable*** from a vertex v if the following procedure always give an Eulerian trial.

Start at v and traversing any incident edge . On arriving at a vertex u, depart through any incident edge not yet covered and continue until all the edges are covered.

If a graph is arbitrarily traversable from a vertex then it is obviously Eulerian.

The following theorem due to Ore (1951) tells just when a given graph is arbitrarily traversable from a chosen point.

Theorem 4.3: [*Ore's theorem*]

An Eulerian graph G is arbitrarily traversable from a vertex v in G iff every cycle in G contains v.

Fleury's Algorithm :

Step – 1:

Choose an arbitrary vertex v_0 and set walk $W_0 = v_0$.

Step – 2:

Suppose that the trial $W_i = v_0\ e_1\ v_2\ e_2\ \ldots\ e_i\ v_i$ has been chosen. Then Choose an edge e_{i+1} from $X(G) - \{e_1, e_2, \ldots, e_i\}$ in such a way that

(i). e_{i+1} is incident with v_i .

(ii).Unless there is no alternative, e_{i+1} is not a bridge of G -{e_1, e_2,...,e_i}.

Step – 3:

Stop when step – 2 can no longer be implemented.

Note: Fleury"s algorithm construct a trial in G.

If G is Eulerian then any trial in G constructed by Fleury"s algorithm is an Euerian trial in G.

***Question* 1**: For what value of n , K_n is Eulerian ?

Answer: K_n is Eulerian when n is odd .

***Question 2*:** For what value of m and n, $K_{m,n}$ is Eulerian ?

Answer: $K_{m,n}$ is Eulerian when both m and n are even.

HAMILTONIAN GRAPHS

In 1959, Sir William Hamilton devised a mathematical game on the graph of the dodecahedron (Fig. 2.10).

Definition: A spanning cycle in a graph is called a ***Hamiltonian cycle***.

A graph having a Hamiltonian cycle is called a ***Hamiltonian graph***.

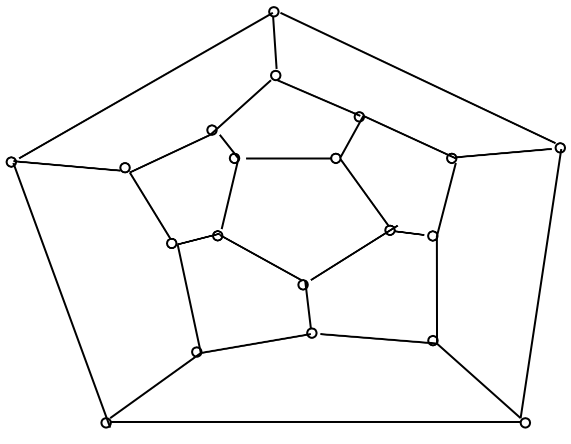


Fig. 4.4

Definition: A block with two non adjacent vertices of degree 3 and all other vertices of degree 2 is called a ***theta graph***.

Example:

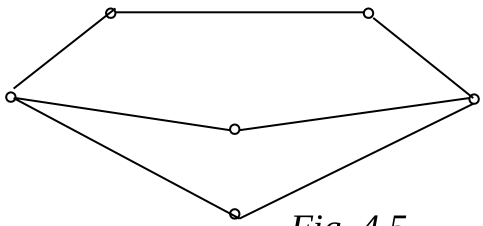


Fig. 4.5

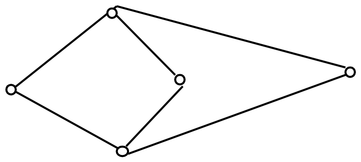


Fig. 4.6

The graphs given in the Fig. 4.5 and Fig. 4.6 are theta graphs.

Note:

A theta graph is non – Hamiltonian and every non – Hamiltonian 2 – connected graph has a theta sub graph.

Theorem 4.4: Every Hamiltonian graph is 2 – connected.

Proof:

Let G be a Hamiltonian graph.

Then G has a Hamiltonian cycle Z.

For any vertex v of G, $Z - v$ is connected.

$\therefore$ $G - v$ is connected.

i.e. any vertex in G will not be a cut point.

Hence the minimum number of points whose removal gives a disconnected or trivial graph will be ≥ 2. i.e. $\kappa \geq 2$.

Thus G is 2 – connected.

Theorem 4.5: If G is Hamiltonian then for every non – empty proper subset S of V(G), ω (G – S) ≤ |S| where ω (H) denotes the number of components in any graph H.

Proof:

Let G be a Hamiltonian graph.

Then G has a Hamiltonian cycle Z.

Let S be any non – empty proper subset of V(G).

Clearly ω (Z – S) ≤ |S|

Also Z – S is a spanning sub graph of G – S.

$\therefore \quad \omega(G - S) \leq \omega(Z - S)$

$\leq |S|$.

Hence $\omega(G - S) \leq |S|$.

Example:

K_n is Hamiltonian for all *n*.

$K_{m, n}$ is Hamiltonian if m = n.

When m < n, $K_{m, n}$ is non Hamiltonian.

Note :

1) The above theorem is useful in showing that some graphs are non Hamiltonian.
2) The converse of the above theorem is not true. For example, the Peterson graph (*Ref.* Fig. 1.4) satisfies the conditions of the theorem but is non Hamiltonian.

Theorem 4.6: [Dirac Theorem , 1952] *(Sufficient condition for a graphG to be Hamiltonian).*

If G is a graph with $p \geq 3$ vertices and $\delta \geq p/2$ then G is Hamiltonian.

Proof:

Let G be a graph with $p \geq 3$ vertices and $\delta \geq p/2$.

Suppose the theorem is false.

Let G be a maximal (with respect to number of edges) non

Hamiltonian graph with p vertices..

Since $p \geq 3$, G is not complete.

$\therefore$ There exists two non adjacent vertices in G.

Let u and v be the non adjacent vertices in G.

Then G + uv is Hamiltonian.

Since G is non Hamiltonian, each Hamiltonian cycle of G + uv

must contain the line uv.

$\therefore$ G has a spanning path $v_1, v_2, \ldots, v_p$ with origin $u = v_1$ and terminus

$v = v_p$.

Let $S = \{v_i / u v_{i+1} \in E\}$ and $T = \{v_i / i < p, v_i v \in E\}$, E is the edge set

of G.

Clearly $v_p \notin S \cup T$

$\therefore |S \cup T| < p$...(1)

To prove $S \cap T = \varphi$

Suppose $S \cap T \neq \varphi$. Then there exists at least one vertex $v_i \in S \cap T$.

$\therefore u v_{i+1} \in E$ and $v_i v \in E$

Then $v_1, v_2, \ldots, v_i, v_p, v_{p-1}, \ldots, v_{i+1}, v_1$ is a Hamiltonian cycle in G.

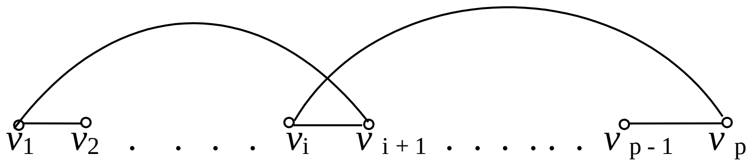


Fig. 4.6

This is a contradiction to G in non Hamiltonian.

$\therefore S \cap T = \varphi \Rightarrow |S \cap T| = 0$.

Hence (1) becomes, $|S| + |T| < p$............................ (2)

Also by the definition of S and T,

$|S| = d(u)$ and $|T| = d(v)$.

But, $d(u) \geq \delta \geq p/2$ and $d(v) \geq \delta \geq p/2$

$\therefore d(u) + d(v) \geq p \Rightarrow |S| + |T| \geq p$.

This is a contradiction to equation (2).

Hence G is Hamiltonian.

Theorem 4.7: Let G be a graph with p points and let u and v be nonadjacent points in G such that $d(u) + d(v) \geq p$. Then G is Hamiltonian iff $G + uv$ is Hamiltonian.

Proof:

Let G be a graph with p points and let u and v be nonadjacent points in G such that $d(u) + d(v) \geq p$..........................(1)

Assume that G is Hamiltonian.

To prove that $G + uv$ is Hamiltonian.

Since G is Hamiltonian, G has a Hamiltonian cycle Z.

This cycle Z is a Hamiltonian cycle in $G + uv$.

$\therefore$ $G + uv$ is Hamiltonian.

Conversely, assume that $G + uv$ is Hamiltonian.

To prove that G is Hamiltonian.

Suppose G is non Hamiltonian.

Let $S = \{v_I \,/\, u\, v_{i+1} \in E\}$ and

$T = \{v_I \,/\, i < p,\ v_i v \in E\}$, where E is the edge set of G.

Let $v_1, v_2, \ldots, v_p$ be a spanning path in G with origin $u = v_1$ and terminus $v = v_p$.

Clearly v_p is not an element of S U T.

$\therefore |S \cup T| < p$...(2)

Also $S \cap T = \varphi$ and $|S| = d(u)$; $|T| = d(v)$

$\therefore$ (2) $\Rightarrow |S| + |T| < p$.

i.e. $d(u) + d(v) < p$.

This is a contradiction to (1).

$\therefore$ G is Hamiltonian.

CLOSURE OF A GRAPH

Definition: The closure of a graph G with p points is the graph obtained from G by repeatedly joining pairs of non - adjacent vertices whose degree sum is at least p until no such pair remains. The closure of G is denoted by c(G).

Theorem 4.8: c (G) is well defined.

Proof:

Let G be a graph with p vertices.

Let G_1 and G_2 be two graphs obtained from G by repeatedly joining pairs of non – adjacent vertices whose degree sum is p until no such pairs remains.

Let $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ be the sequences of edges added to G in obtaining G_1 and G_2 respectively.

To prove that $\{x_1, x_2, \ldots, x_m\} = \{y_1, y_2, \ldots, y_n\}$.

If possible, let $x_{i+1} = uv$ be the first edge in the sequence

$\{x_1, x_2, \ldots, x_m\}$ that is not an edge of G_2.

Let $H = G + \{x_1, x_2, \ldots, x_i\}$.

Since $x_{i+1} = uv$ is the next edge to be added to H in the process of constructing G_1, we have

$$d_H(u) + d_H(v) \geq p .$$

Also, H is a sub graph of G_2.

$$\therefore d'(u) \geq d_H(u) \quad \text{and } d'(v) \geq d_H(v) ,$$

where $d'(u)$ and $d'(v)$ denote degrees of u and v in G_2.

Hence $d'(u) + d'(v) \geq d_H(u) + d_H(v)$

$$\geq p .$$

$\therefore\ x_{i+1} = u\,v$ is the next edge to be added to H to get G_2.

Hence each x_i is an edge of G_2.

Similarly we can prove that each y_i is an edge of G_1.

$\therefore\ \{ x_1, x_2, \ldots, x_m\} = \{ y_1, y_2, \ldots, y_n \}$.

i.e. $G_1 = G_2$.

Hence c (G) is well defined.

Theorem 4.9: A graph is Hamiltonian iff its closure is Hamiltonian.

Proof:

Let $x_1, x_2, \ldots, x_n$ be the sequences of edges added to G to get the closure of G.

Let $G_1, G_2, \ldots, G_n = c(G)$ be the successive graphs obtained.

Applying the theorem 2.20 repeatedly, we get

G is Hamiltonian iff G_1 is Hamiltonian

iff G_2 is Hamiltonian

.

.

.

iff $G_n = c(G)$ is Hamiltonian.

Corollary: Let G be a graph with at least 3 points. If c (G) is complete then G is Hamiltonian.

Theorem 4.10: [Chavatal theorem , 1972]

Let G be a graph with degree sequence $(d_1, d_2, \ldots, d_p)$ where $d_1 \leq d_2 \leq \ldots \leq d_p$ and $p \geq 3$. Suppose that for every value of $m < \frac{p}{2}$, $d_m > m$ or $d_{p-m} \geq p - m$.

(i.e. there is no value of $m < \frac{p}{2}$ for which $d_m \leq m$ or $d_{p-m} < p - m$).

Then G is Hamiltonian.

Proof:

Let G be a graph with degree sequence $(d_1, d_2, \ldots, d_p)$ where $d_1 \leq d_2 \leq \ldots \leq d_p$ and $p \geq 3$.

Suppose that there is no value of $m < \frac{p}{2}$, $d_m \leq m$ or $d_{p-m} < p - m$.

To prove G is Hamiltonian.

i.e. to prove $c(G)$ is complete.

Suppose $c(G)$ is not complete.

Then there exist at least two non – adjacent vertices.

Let u and v be two non – adjacent vertices in $c(G)$ with $d'(u) \leq d'(v)$ and $d'(u) + d'(v)$ is maximum, where $d'(u)$ denote the degree of vertex v in $c(G)$.

Let $d'(u) = m$.

Here, u and v are not adjacent.

$\therefore d'(u) + d'(v) < p$.

$\therefore\ d'(v) < p - m$.

We have $d'(u) \leq d'(v) < p - m$.

i.e. $m < p - m$.

i.e. $m < \frac{p}{2}$

$\therefore$ There is a value of m less than $\frac{p}{2}$.

Let S denote the set of vertices in V – {v} which are not adjacent to v in c (G).

Let T denote the set of vertices in V – {u} which are not adjacent to u in c (G).

Clearly, $|S| = p - 1 - d'(v)$ and $|T| = p - 1 - d'(u)$

i.e. $|S| \geq p - 1 - (p - m)$, [Since $d'(v) < p - m$]

i.e. $|S| \geq m - 1$

$\therefore\ |S| > m$

i.e. c (G) has at least m points with degree $\leq m$.

Also, each vertex in T U {u} has degree $\leq p - m$.

$\therefore\ |T| = p - 1 - m$ and $|T \cup \{u\}| = p - m$.

i.e. c (G) has at least $p - m$ vertices of degree $\leq p - m$.

Because G is a spanning sub graph of c (G), degree of each point in G cannot exceed that in c (G).

Hence G satisfies the condition that there is a value of $m < \frac{p}{2}$, $d_m \leq m$

and $d_{p-m} < p - m$.

This is a contradiction to the hypothesis..

$\therefore$ c (G) is complete.

Hence G is Hamiltonian.

Problem 1: Show that the Peterson graph is Hamiltonian.

Solution: Consider the Fig. 2. 14.

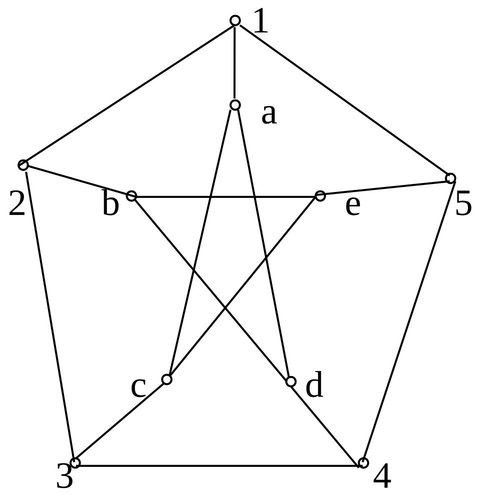


Fig. 4.7

Let us label the vertices as in Fig. 4.7.

We know that, " A regular spanning sub graph of degree 1 is called a one – factor ".

If the Peterson has a Hamilton cycle C then G – E (C) must be a regular spanning sub graph of degree 1.

Let us find all 1 – factors in G and show that they are from the Hamiltonian cycle of G.

Case (1):

Consider the subset A = { 1 a, 2 b, 3 c, 4 d, 5 e} of the edge set of G.

Clearly A is a 1 – factor of G.

But G – A is the union of two disjoint cycles and hence is not a Hamiltonian cycle of G.

Case (2):

If the 1 - factor contains 4- edges from A then the only line passing through the remaining two points must also be included in the one – factor .

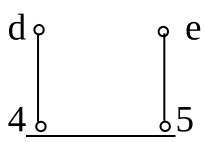
So, again we get A.

Case (3):

If the 1 - factor contains just 3- edges from A then two choices can be made.

Choice -1:

Let the 1 – factor contains the edges 1 a , 2 b and 3 c . Now, the sub graph induced by the remaining 4 points is the path

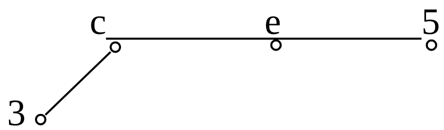


The unique 1 – factor in this path is {4 d , 5 e}.

Thus the 1 – factor of G considered becomes A.

Choice -2:

Let the 1 – factor contains the edges 1a , 2b and 4d . The remaining 4 points is the path



The unique 1 – factor in this path is {3 c , 5 e}.

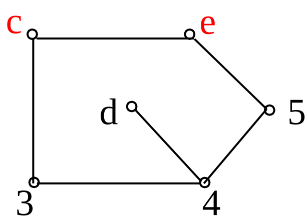
Thus the 1 – factor of G considered becomes A.

Case (4):

If a 1 - factor contains just 2- edges from A then again two choices are possible.

Choice -1:

Let the 1 – factor contain the edges 1 a and 2 b. Now, the sub graph induced by the remaining 6 points gives the path



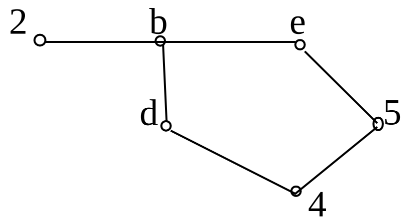
Here the d point has degree 1.

∴ Any 1 – factor of this sub graph must contain the edge 4 d .

Thus case (3) is repeated.

Choice -2:

Let the 1 – factor contain the edges 1 a and 3 c. Now, the sub graph induced by the remaining 6 points gives the path.



Here, the point 2 has degree 1.

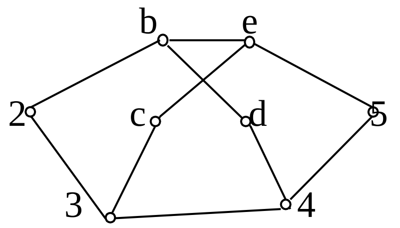
∴ Any 1 – factor of this sub graph must contain the edge 2 b .

Thus case (3) is repeated.

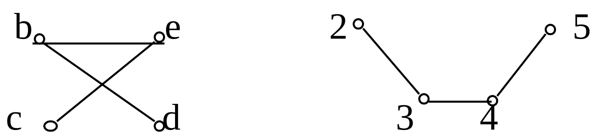
Case (5):

Let the 1 – factor contain just one edge of A, say 1a.

The induced sub graph by the remaining 8 points is



i.e.



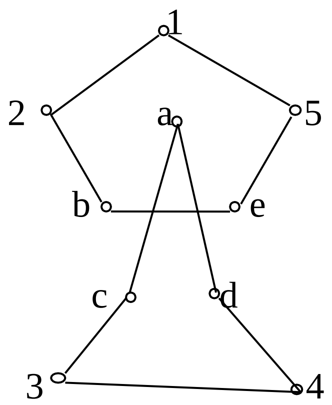
This contains two different paths c e b d and 2345 each of length 3.

Here, degree of c = 1 = degree of d

∴ The 1 – factor must contain the edges 2 3 and 4 5.

Hence the $1 -$ factor is { 1 a, c e , b d, 23, 45}.

Now, $G - B$ is



It is the union of two disjoint cycles and hence it is not a Hamiltonian.

Case (6):

Suppose there exist a $1 -$ factor that does not contain any edge from A.

It can contain at most 2 –edges from the cycle 123451 and at most 2 - edges from the cycle a c e b d a.

Hence it can contain at most $4 -$ edges.

Hence there does not exist such a $1 -$ factor.

From the above six cases, G has no Hamiltonian cycle.

$\therefore$ G is non Hamiltonian.

Exercises:

1). Give an example of a Hamiltonian graph G that contains an induced sub graph isomorphic to the graph in Fig . 2.11.

2). Give an example of a Hamiltonian graph G such that c (G) is not Complete.

3). Find the closure of $C_5 + x$ and $K_4 - x$.

Unit V

TREES

Characterisation of trees- center of tree- planarity – definition and properties – characterization of planar graphs- thickness, crossing and outer planarity.

The Concept of a tree was discovered by Cayley in the year 1857.

Definition:

A graph which contains no cycles is called an ***acyclic graph***.

A connected acyclic graph is called a ***tree***.

Note:

- Any graph without cycles is also called a ***forest.***
- Components of a forest are trees.

Example:

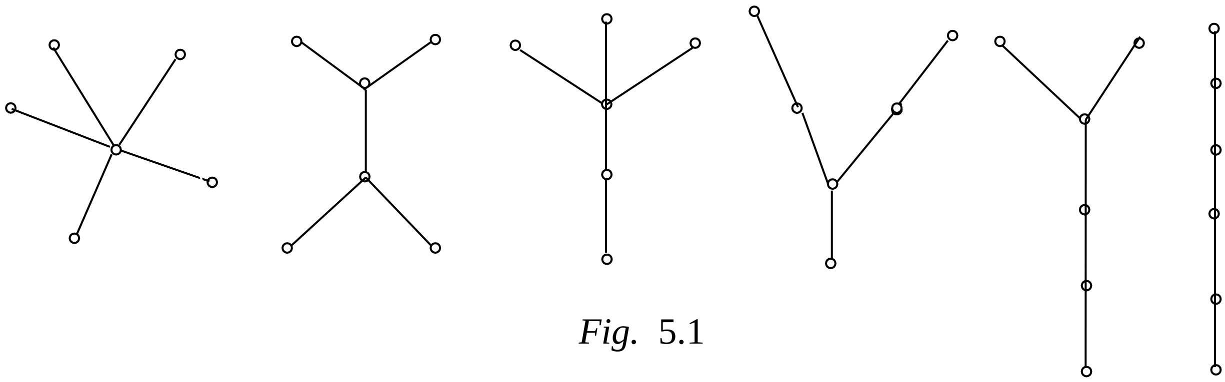


Fig. 5.1

All trees with six vertices is given in Fig. 5.1

CHARACTERISATION OF TREES

Theorem 5.1: Let G be a (*p*, *q*) graph. The following statements are equivalent.

1. G is a tree.
2. Every two points of G are joined by a unique path.
3. G is connected and p = q + 1.
4. G is a cyclic and p = q + 1.

Proof:

Let G be a (p, q) graph .
To prove : (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).(i). To prove : (1) $\Rightarrow$ (2)
Assume that G is a tree.

$\therefore$ G is a connected acyclic graph.

To prove that any two points of G are connected by a unique path.

Let u and v be any two points of G.

Since G is connected there exists a $u - v$ path in G.

Suppose there exists two distinct $u - v$ paths.

$P_1 : u = v_0, v_1, \ldots, v_n = v$ and

$P_2 : u = w_0, w_1, \ldots, w_m = v$.

Let i be the least positive integer such that $1 \leq i < m$ and $w_i \notin P_1$.

$\therefore w_{i-1} \in P_1 \cap P_2$.

Let j be the least positive integer such that $i < j \leq m$ and $w_j \in P_1$.

Then the $w_{i-1} - w_j$ path along P_2 followed by the $w_j - w_{i-1}$ path along P_1 form a cycle. This is a contradiction to G is acyclic.

Hence every two points of G are joined by a unique path.

(ii). To prove: (2) $\Rightarrow$ (3).

Assume that every two points of G are joined by a unique path.

To prove that G is connected and $p = q + 1$.

Clearly, G is connected.

To prove: $p = q + 1$.

Let us prove the result by using induction on p.

When $p = 1$ or $p = 2$; $q = 0$ or $q = 1$

$\therefore$ The result is true when $p = 1$ or $p = 2$

Assume the result for all graphs with less than p points.

To prove the result for a graph G with p points.

Let u and v be any two points of G.

Then by the assumption there exists a unique u - v path in G.

Consider any line x on the path. Then G - x is a disconnected graph with exactly two components G_1 and G_2.

Let G_1 be a (p_1, q_1) graph and G_2 be a (p_2, q_2) graph.

Then $p_1 + p_2 = p$ and $q_1 + q_2 = q - 1$.

Clearly, p_1 and $p_2 < p$.

$\therefore$ By the induction assumption, $p_1 = q_1 + 1$ and $p_2 = q_2 + 1$.

Now, $p = p_1 + p_2$

$= q_1 + q_2 + 2.$

$= q - 1 + 2$

$= q + 1.$

Hence G is connected and $p = q + 1$.

(iii). To prove: (3) $\Rightarrow$ (4).

Assume that G is connected and $p = q + 1$.

To prove that Gis acyclic and $p = q + 1$.

Suppose that G contains a cycle of length n.

There are n points and n lines on this cycle.

Fix a point u on the cycle. Consider any one of the remaining

$p - n$ points not on the cycle, say v .

Since G is connected we can find a shortest $u - v$ path in G.

Consider the line on this shortest path incident with v.

The $p - n$ lines thus obtained are all distinct.

$$\therefore \quad q \geq (p - n) + n$$

$$\Rightarrow \quad q \geq p$$

This is a contradiction to $p = q + 1$.

Hence G is acyclic and $p = q + 1$.

(iii). To prove: (4) $\Rightarrow$ (1).

Assume that G is acyclic and $p = q + 1$.

To prove that G is a tree.

It is enough to prove that G is connected.

Suppose G is not connected. Then G has more than one component.

Let $G_1, G_2, \ldots, G_k$, $k \geq 2$ be the components of G.

Since G is acyclic, each of these components is a tree.

$\therefore\ p_i = q_{i+1}$ for i = 1 , 2, . . . , k , where G_i is a (p_i, q_i) graph.

$\therefore \sum_{i=1}^{k} p_i = \sum_{i=1}^{k} q_i + 1$.

$\Rightarrow\ p = q + k \geq q + 2$, since $k \geq 2$.

This is a contradiction to $p = q + 1$.

$\therefore$ G is connected.

Hence G is a tree.

Corollary: Every non – trivial tree G has at least two vertices of degree 1.

Proof:

Since G is non – trivial, $d(v) \geq 1$ for all points v.

Also G is a tree.

$\therefore\ p = q + 1$,

Now, $\sum d(v) = 2q = 2(p - 1) = 2p - 2$.

$\Rightarrow d(v) = 1$ for at least two vertices.

Theorem 5.2: Every connected graph has a spanning tree.

Proof:

Let G be a connected graph.

Let T be a minimal connected spanning sub graph of G.

To prove that T is a tree .

By the definition of T , for any line x of T, T – x is disconnected.

$\therefore\ x$ is a bridge of T.

We know that, " A line x of a connected graph G is a bridge iff x is not on any cycle of G".

$\therefore$ x is not on any cycle of T.

$\therefore$ T is acyclic.

Further, T is connected.

$\therefore$ T is a tree .

Hence T is a spanning tree of G.

Corollary: Let G be a (p, q) connected graph . Then $q \geq p - 1$.

Proof:

Let G be a (p, q) connected graph .

We know that, " Every connected graph has a spanning tree".

$\therefore$ G has a spanning tree T.

$\therefore$ T has p points and $p - 1$ lines.

Hence $q \geq p - 1$.

Theorem 5.3: Let T be a spanning tree of a connected graph G. Let $x = uv$ be an edge of G not in T. Then $T + x$ contains a unique cycle .

Proof:

Let T be a spanning tree of a connected graph G.

Also $x = uv$ be an edge of G not in T.

We have T is acyclic.

∴ Every cycle in T + x must contain the edge x.

Hence there exists a one to one correspondence between the cycles in T + x and $u - v$ path in tree T.

We know that , In a tree there is a unique $u - v$ path.

Hence there is a unique cycle in T + x.

CENTRE OF A TREE

Definition: Let v be a point in a connected graph G. The ***eccentricity*** $e(v)$ is defined by $e(v) = \max\{ d(u, v) / u \in V(G)\}$

The ***radius*** r(G) is defined by $r(G) = \min\{ e(v) / v \in V(G)\}$.

v is called a ***central point*** if $e(v) = r(G)$ and the set of all central points is called the ***centre*** of G.

Example: Consider the graph given in Fig. 5.2.

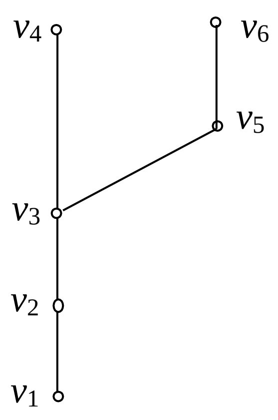


Fig. 5.2

The eccentricity

$e(v_1) = 4$; $e(v_4) = 3$

$e(v_2) = 3$; $e(v_5) = 3$

$e(v_3) = 2$; $e(v_6) = 4$

The radius $r(G) = \min\{ e(v) \ / \ v \in V(G)\}$.

$= 2$

Centre of G $= \{ v \ / \ e(v) = r(G) \}$

$= v_3$.

Theorem 5.4: Every tree has a centre consisting of either one point or two adjacent points.

Proof:

The result is obvious for the trees K_1 and K_2.

Let T be any tree with p ≥ 2 points.

Then T has at least two end points and the maximum distance from a given point u to any other point v occurs only when v is an end point.

Now delete all the end points from T.

The resulting graph T' is also a tree and also the eccentricity of each point in T' is exactly one less than the eccentricity of that vertex in T.

$\therefore$ T and T' have the same centre.

The process of removing end points is repeated .

Finally we get a successive trees having the same centre as T.

Hence we obtain a tree which is either K_1 or K_2.

$\therefore$ The centre of T consists of either one point or two adjacent points

Exercises:

1. Draw all trees with 4 and 5 vertices.
2. Prove that if G is a forest with p points and k components then G has $p - k$ points.
3. Prove that the origin and terminus of a longest path in a tree have degree 1.
4. Show that every tree with exactly 2 vertices of degree 1 is a path.

PLANAR GRAPH AND THEIR PROPERTIES

Definition:

A Graph is said to be ***embedded*** in a surface S if it is drawn on the surface S such that no two edges intersect (cross over).

A graph is called ***planar*** if it can be drawn on a plane without intersecting edges.

A graph is called ***non – planar*** if it is not planar.

A graph that is drawn on a plane without intersecting edges is called a ***plane graph***.

Examples:

(1). K_4 is a plane graph.

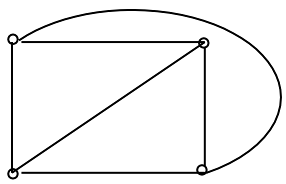


Fig. 5.2

(2). $K_{2,3}$ is a plane graph or planar graph.

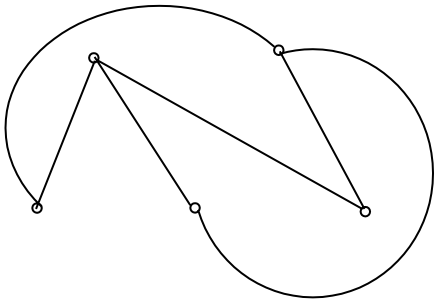


*Fig.*5.3

(3). The graph given in Fig. 5.4 (a) is planar even though it is not plane.

(a) *Fig.* 5.4 (b)

Theorem 5.5: K_5 is non – planar.

Proof:

We know that K_5 has 5 vertices and $5\,C_2 = 10$ edges.

Let the vertices be v_1, v_2, v_3, v_4, v_5.

Out of these 10 edges , 5 edges form a cycle $v_1, v_2, v_3, v_4, v_5, v_1$.

This cycle divides the plane into two regions namely the interior region and the exterior region.

The remaining 5 edges should be drawn either in the interior or in the exterior.

Suppose the edges $v_5 v_3$ and $v_1 v_3$ can be drawn in the interior region without cross over.

The edges $v_4 v_1$ and $v_4 v_2$ can be drawn in the exterior region without cross over.

Now the edge $v_2 v_5$ remains which cannot be drawn without cross over.

Hence K_5 is non – planar.

Definition: Let G be a graph embedded on a plane π. Then π - G is the union of disjoint regions. Such regions are called ***faces*** of G.

Each plane graph has exactly one unbounded face and it is called the ***exterior face***. The interior faces are bounded by cycles.

Theorem 5.6: A graph can be embedded in the surface of a sphere iff it can be embedded in a plane.

Proof:

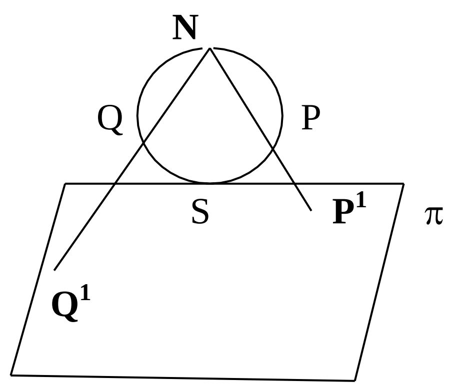


Fig5.5

Let G be a graph embedded on a sphere. Place the sphere on a plane π. Let S be the point of contact of the sphere with the plane.

Draw a normal to the plane and the normal intersects the surface of the sphere at N.

Assume that the sphere is placed in such a way that the point N is different from the vertices of G.

For each point P on the surface of the sphere draw the line NP and it meets the plane at P^1.

The point P^1 is called the stereographic projection of P on the plane.

In this way the vertices and edges of G can be projected on the plane which gives an embedding of the graph G in the plane.

$\therefore$ A graph G can be embedded in the plane.

Conversely, Assume that the graph G can be embedded in the plane. The reverse process obviously gives an embedding of the graph in the surface of the sphere.

Theorem 5.7: [***Euler's Polyhedron Formula***]

If G is a connected plane graph having V , E and F as the vertices , edges and faces respectively, then $|V| - |E| + |F| = 2$.

Proof: Let G be a connected plane graph.

We prove the theorem by using induction on the number of edges of G.

If $|E| = 0$ then clearly $G = K_1$, [since G is connected].

$\therefore$ $|V| = 1$ and $|F| = 1$.

Now, $|V| - |E| + |F| = 1 - 0 + 1 = 2$

$\therefore$ The result is true when $|E| = 0$.

Assume the result for all connected plane graphs with $< |E|$ edges.

To prove the result for a graph G with $|E|$ edges.

If G is a tree then $|E| = |V| - 1$, $|F| = 1$.

Now, L. H. S. $= |V| - |E| + |F|$

$= |E| + 1 - |E| + 1$

$= 2$

= R. H. S.

$\therefore$ The result is true for G is a tree.

If G is not a tree then G contains some cycles.

Let x be an edge contained in some cycle of G.

Then $G' = G - x$ is a connected plane graph.

Also, $|V'| = |V|$, $|E^1| = |E| - 1 < |E|$ and $|F^1| = |F| - 1$.

By induction assumption, $|V'| - |E^1| + |F^1| = 2$.

i.e. $|V| - |E| + 1 + |F| - 1 = 2$.

i.e. $|V| - |E| + |F| = 2$.

$\therefore$ The result is true for G is not a tree.

Hence by the induction principle, the result is true for all connected plane graphs.

Corollary 1: If G is a plane (p, q) graph with a r faces and k components then $p - q + r = k + 1$.

Proof:

Consider a plane embedding of G such that the exterior face of each component contains all other components.

Let the i^{th} component be a (p_i, q_i) graph with r_i faces for each i.

Then each component is a connected plane graph.

$\therefore$ By Euler"s Polyhedron formula,

$$p_i - q_i + r_i = 2.$$

$$\therefore \sum_{i=1}^{k} p_i - \sum_{i=1}^{k} q_i + \sum_{i=1}^{k} r_i = \sum_{i=1}^{k} 2 \qquad \ldots\ldots\ldots(1)$$

Also, $\sum_{i=1}^{k} p_i = p$, $\sum_{i=1}^{k} q_i = q$, $\sum_{i=1}^{k} r_i = r + (k-1)$

$\therefore$ $p - q + r + (k-1) = 2k$

i.e. $p - q + r = k + 1$.

Corollary 2: If G is a (p, q) plane graph in which every face is an n cycle then $q = \frac{n(p-2)}{n-2}$.

Proof:

Let G be a (p, q) plane graph in which every face is an n cycle .

$\therefore$ Each edge lies on the boundary of exactly two faces.

Let $f_1, f_2, \ldots, f_r$ be the faces of G.

Then $2q = \sum_{i=1}^{r}$ (number of edges in the boundary of the face f_i).

$= \sum_{i=1}^{r} n$, since each face is an n - cycle

$= n\, r$

$\Rightarrow r = \frac{2}{n}$.

By Euler"s theorem,

$p - q + r = 2.$

i. e. $p - q + \frac{2q}{n} = 2.$

$\Rightarrow q\ (\frac{2}{n} - 1) = 2 - p$

$\Rightarrow q = \frac{(p-2)}{n-2}$.

Corollary 3: In any connected plane (p, q) graph , $p \geq 3$, with r faces then $q \geq \frac{3r}{2}$ and $q \leq 3p - 6$.

Proof:

Let G be any connected plane (p, q) graph with r faces and $p \geq 3$.

***Case* 1:** Let G be a tree.

Then $q = p - 1$ and $r = 1$.

$\therefore p = q + 1$ and $r = 1$.

$p \geq 3 \Rightarrow q + 1 \geq 3$

$\Rightarrow q \geq 2 > \frac{3}{2}$

$\therefore q \geq \frac{3r}{2}$

Also, $p \geq 3 \Rightarrow 2p \geq 6$

i.e. $2p + p \geq 6 + p$

$\Rightarrow 3p - 6 \geq p > p - 1$

i.e. $3p - 6 \geq q$

i.e. $q \leq 3p - 6$.

Hence for a tree, $\frac{3r}{2} \leq q \leq 3p - 6$.

***Case* 2:** Let G have a cycle.

Let $f_1, f_2, \ldots, f_r$ be the faces of G.

We know that, " Each edge lies on the boundary of at most two faces".

$\therefore$ $2q \geq \sum_{i=1}^{r}$ (number of edges in the boundary of the face f_i).

$\geq \sum_{i=1}^{r} 3$, [since each face is bounded by at least 3 edges]

$= 3r$

i.e. $q \geq \frac{3r}{2}$.

By Euler‟s formula, $p - q + r = 2$

i.e. $r = 2 + q - p$.

We have $q \geq \frac{3r}{2}$

$\therefore$ $q \geq \frac{3(2+q-p)}{2}$

i.e. $2q \geq 6 + 3q - 3p$.

i.e. $3p - 6 \geq q$

$\Rightarrow q \leq 3p - 6.$

Hence $\frac{3r}{2} \leq q \leq 3p - 6.$

MAXIMAL PLANAR GRAPH

Definition: A graph is called a ***maximal planar*** if no line can be added toit without losing planarity.

A graph is called a ***triangulated graph*** if each face is a triangle.

In a maximal planar graph, each face is a triangle.

Corollary 4: If G is a maximal planar (p, q) graph then and $q = 3p - 6$.

Corollary 5: If G is a plane connected (p, q) graph without triangle and $p \geq 3$ then $q \leq 2p - 4$.

Proof: Let G be a plane connected (p, q) graph without triangle and $p \geq 3$.

Case* 1*: Let G be a tree.

Then $q = p - 1$.

To prove $q \leq 2p - 4$.

We have $p \geq 3 \Rightarrow p + p \geq 3 + p$

$\Rightarrow 2p \geq 3 + q + 1$

$\Rightarrow q \leq 2p - 4.$

Case* 2*: Let G have a cycle.

$\therefore$ The boundary of each force has at least four edges. Also each edge lies on at most 2 faces.

$\therefore 2q \geq \sum_{i=1}^{r}$ (number of edges in the boundary of the face f_i).

$= \sum_{i=1}^{r} 4$

$= 4r$

i.e. $2q \geq 4r$.

By Euler‟s formula, $p - q + r = 2$

i.e. $r = 2 + q - p$.

$\therefore 2q \geq 4(2 + q - p)$.

$\Rightarrow q \leq 2p - 4$.

Corollary 6: The graphs K_5 and $K_{3,3}$ are not planar.

Proof:

We know that,

"K_5 is a connected (5, 10) graph".

and " In any connected plane (p, q) graph $p \geq 3$, $q \leq 3p - 6$ ".

Here, $3p - 6 = 3 \times 5 - 6 = 9$

i.e. $10 \nleq 9$

$\Rightarrow$ the inequality $q \leq 3p - 6$ is not satisfied.

Hence K_5 is not planar.

Next, To prove $K_{3,3}$ is not planar.

We know that, $K_{3,3}$ is a complete bipartite (6, 9) graph.

Also$K_{3,3}$ has no triangles.

We also know that "If G is a plane connected (p, q) graph without triangle $p \geq 3$ then $q \leq 2p - 4$".

Here $2p - 4 = 2 \times 6 - 4 = 8$

$\therefore 9 \nleq 8 \Rightarrow q \nleq 2p - 4$.

Hence $K_{3,3}$ is not planar.

Corollary 7: Every planar graph G with p ≥ 3 points has at least three points of degree < 6.

Proof:

Let G be a planar graph with p ≥ 3 points.

To prove that G has at least three points of degree < 6.

By corollary 3, $q \leq 3p - 6$

$\therefore 2q \leq 6p - 12$

i.e. $\sum_{i=1}^{p} d_i \leq 6p - 12$ ....(1).

Suppose at most two points have degree < 6.

Also G is connected.

$\therefore d_i \geq 1, \forall$ i.

$\sum_{i=1}^{p} d_i \geq 6 + 6 + \ldots + (p - 2) + 1 + 1$

$= 6(p - 2) + 2$

$= 6p - 10$

i.e. $\sum_{i=1}^{p} d_i \geq 6p - 10$

This is a contradiction to (1).

Hence G has at least three points of degree < 6.

Theorem 5.8: Every polyhedron has at least two faces with the same number of edges on the boundary.

Proof:

Let G be the graph got from the polyhedron.

Then G is planar and 3 – connected.

i.e. $\kappa \geq 3$

$\therefore \delta \geq 3$, [Since $\kappa \leq \lambda \leq \delta$]

We know that " The number of faces adjacent to any given face f is equal to the number of boundary edges of the face f ".

Let $f_1, f_2, \ldots, f_m$ be the faces of the polyhedron and e_i be the boundary edges of the face f_i .

Let the faces be labeled in such a way that $e_i \leq e_{i+1}$ for every i .

To prove that there exists at least two faces with the same number of boundary edges.

Suppose no two faces have the same number of boundary edges.

Then $e_{i+1} - e_i \geq 1$ for every i .

$\therefore \sum_{i=1}^{m-1}(e_{i+1} - e_i) \geq \sum_{i=1}^{m-1} 1 = m - 1.$

Also, $\sum_{i=1}^{m-1}(e_{i+1} - e_i) = (e_2 - e_1) + (e_3 - e_2) + \ldots + (e_m - e_{m-1})$

$= e_m - e_1$

i.e. $e_m - e_1 \geq m - 1.$

i. e. $e_m \geq (m-1) + e_1$

i.e. $e_m \geq m + 2$, [since $e_1 \geq 3$]

i.e. The m th face is adjacent to at least $(m + 2)$ faces.

This is a contradiction to the fact that there are only m faces.

Hence there exists at least two faces with the same number of boundary edges.

CHARACTERISATION OF PLANAR GRAPH

(1). A graph is planar iff all its blocks are planar.

(2). A disconnect graph is planar iff each of its components are planar.

(3). Every subgraph of a planar graph is planar.

Definition: Let $x = u\,v$ be an edge of a graph G. The line x is said to be subdivided when a new point w is adjoined to G and the line x isreplaced by the lines $u\,w$ and $w\,v$. This process is also called an ***elementary subdivision*** of the edge x.

Two graphs are called ***homeomorphic*** if both can be obtained from the same graph by a sequence of subdivisions of the lines.

Solved Problem:

Problem 1: If a (p_1, q_1) graph and a (p_2, q_2) graph are homeomorphic then $p_1 + q_2 = p_2 + q_1$.

Solution:

Assume that the graphs $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are homeomorphic.

Then G_1 and G_2 can be got from a (p, q) graph G by a series of elementary subdivisions.

Let G_1 can be got from G by r elementary subdivisions and G_2 can be got from G by s elementary subdivisions.

In each elementary subdivision , the number of points as well as the number of lines increases by one.

$\therefore p_1 = p + r$, $q_1 = q + r$; $p_2 = p + s$, $q_2 = q + s$.

L. H. S. $= p_1 + q_2 = p + r + q + s$

$= (p + s) + (q + r)$

$= p_2 + q_1$

= R. H. S.

Theorem 5.9: [Kuratowski Theorem, 1930]

A graph is planar iff it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

Note:

1). The above Kuratowski theorem gives the *necessary and sufficient condition for a graph to be planar*.

2). The graphs K_5 or $K_{3,3}$ are called ***Kuratowski graphs***.

THICKNESS, CROSSING AND OUTER PLANARITY

Definition: The ***crossing number*** of a graph G is the minimum number of pair wise intersections of the edges when G is drawn in the plane.

The crossing number of a planar graph is zero.

The crossing number of each of the Kuratowski graphs is one.

Definition: A planar graph is called ***outer planar*** if it can be embedded in the plane so that all its vertices lie on the same face. This face is often chosen to be the exterior face.

Definition: The outer planar graph is called ***maximal outer planar*** if no line can **b**e added without losing outer planarity.

Every maximal outer planar graph is a triangularisation of a polygon. But, every maximal plane graph is a triangularisation of the sphere.

Definition: The ***genus*** of a graph G is defined to be the minimum Number of handles to be attached to a sphere so that G can be drawn on The resulting surface without intersecting lines.

Every planar graph has genus 0.

K_5 , K_6, K_7, $K_{3,3}$ and $K_{4,4}$ each has genus 1.